Leverage Dynamics without Commitment

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We develop a general methodology to characterize equilibrium leverage dynamics in a tradeoff model when the firm can continuously adjust leverage and cannot commit to a policy ex ante. While the leverage ratchet effect leads shareholders to increase debt gradually over time, asset growth and debt maturity cause leverage to mean-revert slowly towards a target. Because investors anticipate future debt issuance, credit spreads are increased, offsetting fully the tax benefits of future debt issuance. Finally, although the target leverage and speed of adjustment depend critically on debt maturity, shareholders are indifferent toward the debt maturity structure.

Keywords: Capital Structure, Target Leverage, Tradeoff Theory, Credit Spreads, Debt Maturity, Debt Overhang, Dynamic Games, Coase Conjecture

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1. Introduction

Understanding the determinants of a firm's capital structure, and how its leverage is likely to evolve over time, is one of the central questions in corporate finance. Leverage and its expected dynamics are crucial to valuing the firm, assessing its credit risk, and pricing its financial claims. Forecasting the optimal response of leverage to shocks, such as the 2007-2008 financial crisis, is necessary to anticipate the likely consequences of a crisis and its aftermath, and to evaluate alternative policy responses.

Despite its importance, a fully satisfactory theory of leverage dynamics has yet to be found. Many models assume the absolute level of debt is fixed; for example, in the traditional framework of Merton (1974), as well as Leland (1994, 1998), the firm is committed not to change its outstanding debt before maturity, irrespective of the evolution of the firm's fundamentals. As a result, the dynamics of firm leverage is driven solely by the stochastic growth in value of the firm's assets-in-place. More recent work that allows the firm to restructure its debt over time typically assumes that all existing debt must be retired (at a cost) before any new debt can be put in place.[1] These assumptions are neither innocuous, as the constraints on leverage generally bind in the model, nor are they consistent with practice, where firms often borrow incrementally over time. See, for example, Figure 1, which shows how debt levels for American and United Airlines changed over time in response to fluctuations in their enterprise values (market value of equity plus book value of debt).

In contrast, we study a model in which equity holders lack the ability to commit to their future leverage choices and can issue or buyback debt at the current market price at any time. Aside from corporate taxes and bankruptcy costs, there are no other frictions or transactions costs in our model. In such a setting, when debt can be freely adjusted over time, it is feasible for the firm to avoid the standard leverage "tradeoff" by simply increasing debt to exploit tax shields when cash flows are high and reducing debt to avoid distress costs when cash flows fall.

[1] See e.g. Fischer, Heinkel, Zechner (1989), Titman and Tsyplakov (2007), Goldstein, Ju, and Leland (2001) and Strebulaev (2007), Dangl and Zechner (2016).

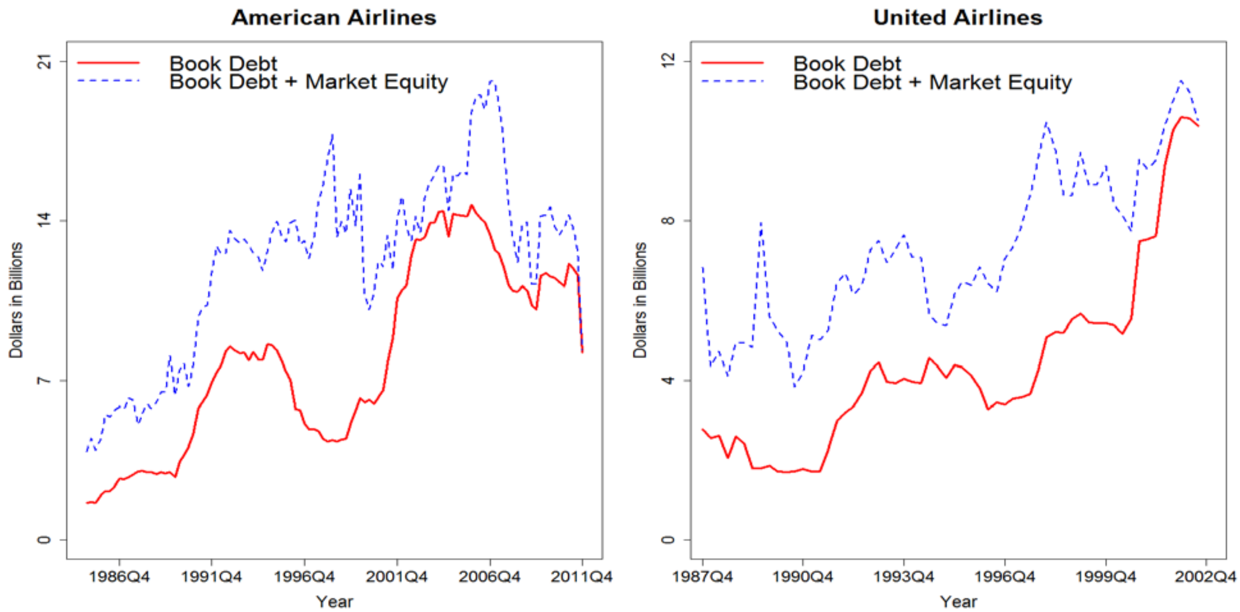

Figure 1. Time-series of outstanding book debt and enterprise value for American and United Airlines, for fifteen years before their bankruptcies in 2011 and 2002, respectively. Book debt is calculated as the sum of "long-term debt" and "debt in current liabilities", and market equity is calculated as "stock price" multiplying "common shares outstanding." Data source: WRDS.

But although such an ideal policy is feasible, absent commitment an important agency friction emerges with regard to the firm's future leverage choices. As emphasized by Admati et al. (2017), equity holders will adjust leverage to maximize the current share price rather than total firm value. They demonstrate a "leverage ratchet effect," in which equity holders are never willing to voluntarily reduce leverage, but always have an incentive to borrow more -- even if current leverage is excessive and even if new debt must be junior to existing claims. While the leverage ratchet effect is itself quite general, they numerically calculate a dynamic equilibrium only for a specialized model in which debt is perpetual and the firm does not grow but is subject to Poisson shocks.

Solving the dynamic tradeoff model without commitment is challenging because of the dynamic interdependence of competitive debt prices today and equity's equilibrium leverage/default policies in the future. In this paper we develop a methodology to solve for such an equilibrium in a general setting that allows for finite maturity debt, asset growth, investment, and both Brownian and Poisson shocks. In this equilibrium, equity holders increase debt gradually over time, at a rate which increases with the current profitability of the firm. On the other hand, following negative shocks, equity holders never voluntarily reduce leverage, but do allow it to decline passively via debt maturity and asset growth.

In our model, equity holders keep issuing debt to exploit tax benefits even after the firm's leverage passes above the "optimal" level with commitment, leading to excessive inefficient default. This result holds even when there is *no* dilution motive to issue debt (either because there is zero recovery value in bankruptcy, or debt is prioritized so that newly issued debt must be junior to all existing debt). However, even without a direct dilution effect, there is an indirect "dilution" or devaluation effect associated with new debt issuance, as additional leverage raises the probability of default for all debt holders. Creditors anticipate the devaluation associated with future over-borrowing, and consequently lower the price they will pay for the debt. This price impact is enough to offset the tax advantage of leverage so that, on the margin, equity holders are indifferent to leverage increases. As a result, equity holders obtain the same value in equilibrium *as if* they commit not to issue any debt in the future. In other words, the extra tax shield benefits that tempt equity holders are exactly dissipated by the bankruptcy costs caused by excessive leverage.

We apply our methodology to the special case of geometric Brownian motion (as in Leland (1994)) and solve for the equilibrium debt price and issuance policy in closed form. Because equity holders refuse to buy back debt once it is issued, debt issuance becomes effectively irreversible, slowing its initial adoption. Debt accumulates over time at a rate that increases with profitability, while if profits decline sufficiently, new issuance drops below the rate of debt maturity and the debt level falls. Leverage is thus path dependent, and we show explicitly that the firm's outstanding debt at any point of time can be expressed in terms of a weighted-average of the firm's past earnings. The endogenous adjustment of leverage leads the firm's interest coverage ratio to mean revert gradually in equilibrium, with the speed of adjustment decreasing with debt maturity and asset volatility. These dynamics differ from the abrupt adjustment to a "target" leverage level implied by models with an exogenous adjustment cost (for instance, Fischer, Heinkel, Zechner, 1989; Goldstein, Ju, Leland, 2001; and Strebulaev, 2007; etc.).

We compare our model without commitment to standard full commitment benchmarks. While equity prices coincide with models in which there is no new debt issuance, bond investors' anticipation of future borrowing causes credit spreads to be bounded away from zero and much larger than in standard models.

We also study the optimal debt maturity structure modelled in terms of a constant required repayment (or amortization) rate stipulated in the debt contract. Our model without commitment provides a fresh perspective on this question. We show that at every point in time, equity holders are indifferent to the maturity choice for future debt issuance. Short maturity debt leads to higher leverage on average, as equity holder issue debt more aggressively knowing leverage can be reversed when debt matures. Nevertheless, the gain from additional tax shields is offset by increased default costs. Thus, the agency costs associated with the leverage ratchet effect persist even for instantaneously maturing debt.

In our model firms with different debt maturity structures can have quite different leverage dynamics; yet firms are indifferent in the debt maturity choice. This provides a potential explanation for the finding in Lemmon, Roberts, and Zender (2008), that much of the cross-sectional variation in firms' capital structure is persistent and largely unexplained by observable characteristics. From the perspective of our model, small perturbations or frictions that may lead firms to pick differing initial maturity structures will lead over time to dramatically different leverage outcomes. [2]

Finally, we consider the interaction of the firm's leverage and investment policies. When the firm cannot commit to its investment policy, leverage distorts investment due to debt overhang. Compared to a fixed-debt policy as in Leland (1998), the no commitment leverage policy leads to less investment (due to debt overhang) when profitability is high, but more investment when profitability is low. Ex ante, shareholders are not incrementally harmed by debt overhang, as underinvestment substitutes for default costs. The firm tends to issue new debt more slowly, and targets a lower level of leverage in the presence of debt overhang. Near default, however, shareholders issue debt more aggressively when they have the option to cut investment.

[2] The choice of debt maturity structure does affect the value of equity if the firm is forced to borrow a fixed amount upfront. Indeed, this question has been studied in the Leland (1998) setting, and often long-term debt, which minimizes rollover risk, is preferred (He and Xiong, 2012; Diamond and He, 2014). In contrast, we show that without commitment, firms prefer short-term debt for any positive targeted debt financing. Shareholders of a firm with shorter-term debt are more willing to allow leverage to decline following negative shocks, and this future equilibrium leverage policy lowers the required default premium today. Of course, longer-term debt is preferred from a social perspective, which lowers expected bankruptcy costs. (Another possible force favoring short-term debt is investors' liquidity preference, which is modeled in He and Milbradt (2014).)

Our paper is most closely related to Admati et al. (2017). They demonstrate the leverage ratchet effect in the context of a one-time leverage adjustment, and then numerically evaluate a dynamic equilibrium in a stationary model with regime shocks and perpetual debt. Our paper studies leverage dynamics in a richer continuous-time framework that allows for both asset growth and debt maturity, as well as both Brownian and Poisson shocks. We develop a general methodology to solve for an important class of equilibria, and for the standard workhorse model of Leland (1994), we solve for the equilibrium in closed-form, allowing for deeper analysis.

In Dangl and Zechner (2016), the firm can choose how much maturing debt to rollover, but covenants place a cap on the rate of new debt issuance that prevent it from increasing the face value of its debt outstanding without first repurchasing all existing debt (at par plus a call premium and a proportional transaction cost). Rolling over debt maintains the firm's tax shields, as in in our model, and directly dilutes current creditors given their setting with a strictly positive recovery rate and pari-passu debt (which we analyze in Section 3.4). They show that when debt maturity is long, equity holders will rollover existing debt fully as it comes due, except for when leverage is so low that recapitalization to a higher face value of debt is imminent (in which case it is not worthwhile to issue debt that is likely to be replaced soon, at a cost). If debt maturity is sufficiently short, however, then when facing high leverage shareholders may rollover only a portion of the maturing debt so that the total face value of debt gradually declines. This behavior abruptly reverses when the firm approaches default as shareholders maximize dilution (and minimize equity injections) by again rolling over debt fully.[3] Importantly, they show that firm value is not monotonic in debt maturity; depending on parameters, an interior optimal maturity may exist that trades off the transactions costs of debt rollover (which favors long maturities) with the benefit from debt reductions given high leverage (which favors short maturities).[4] As in our model, the choice of debt maturity becomes an important commitment device that allows for future debt reductions in the face of negative shocks.

In a somewhat different context, Brunnermeier and Yogo (2009) stress the advantage of short-term debt in providing the firm with flexibility to adjust debt quickly in the face of shocks to firm value, but that long-term debt is more effective at reducing costs from rollover risk. Abel

[3] In the extension of our model in which we allow for direct dilution, because there is no constraint on the rate of issuance, we show that the debt issuance rate increases only at the moment of default.

[4] The same trade-off would apply in our model if we were to adopt the same assumption on transaction costs.

(2016) considers a dynamic model with investment in which firms adjust leverage by issuing debt with instantaneous maturity. Abel assumes i.i.d. regime shocks to profitability and shows that in response to a shock, (i) shareholders never reduce the amount of debt, and (ii) only firms that are borrowing constrained (i.e. have borrowed an amount equal to 100% of firm value) choose to increase debt.

Our paper proceeds as follows. In Section 2 we introduce a general continuous-time model of the firm and develop our methodology for solving for an equilibrium in which shareholders adjust debt continuously. Section 3 applies our general results to the special case when cash flows are lognormal with possible jumps and derives a closed-form solution for security prices and debt issuance. Section 4 analyzes debt dynamics and shows that the firm gradually adjusts leverage towards a target level. We then evaluate the firm's choice of debt maturity. Section 5 compares our equilibrium with standard benchmarks such as Leland (1998). Section 6 extends the model to include agency costs of investment, and Section 7 concludes.

2. A General Model

We begin by outlining a general jump-diffusion model of cash flows that encompasses typical settings used in the literature. We include both taxes and bankruptcy costs as in a standard tradeoff model. We depart from the existing literature by assuming equity holders can issue or repurchase debt at any time at the current market price, and analyze the optimal no-commitment leverage policy in equilibrium. Clearly, this policy depends on equilibrium debt prices; but equilibrium debt prices depend on the firm's *future* leverage choices, which determines the likelihood of default. Despite this interdependence, we can characterize the time-consistent leverage policy explicitly and show that the rate of debt issuance is determined by the ratio of tax benefits to the price sensitivity of debt to new issues. We also show that equity values can be computed as though the firm committed not to issue new debt.

2.1. The Firm and Its Securities

All agents are risk neutral with an exogenous discount rate of $r>0$.[5] The firm's assets-in-place generate operating cash flow (EBIT) at the rate of Y_t which evolves according to

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dZ_t + \zeta(Y_{t-})dN_t, \tag{1}$$

where the drift $\mu(Y_t)$ and the volatility $\sigma(Y_t)$ are general functions that satisfy standard regularity conditions; dZ_t is the increment of standard Brownian motion; dN_t is Poisson increment with intensity $\lambda(Y_t)>0$; and $\zeta(Y_{t-})$ is the jump size given the Poisson event.[6]

Denote by F_t the aggregate face value of outstanding debt. The constant coupon rate of the debt is $c>0$, so that over $[t,t+dt]$ debt holders receive coupon payments of $cF_t dt$ in total.[7] The firm pays corporate taxes equal to $\pi(Y_t - cF_t)dt$, where $\pi(\cdot)$ is a non-decreasing function of the firm's profit net of interest. When the marginal tax rate is positive ($\pi'>0$, with "prime" indicating derivative), the net after tax cost to the firm of the marginal coupon payment is $1-\pi'$, reflecting the debt tax shield subsidy.

For simplicity, we assume that debt takes the form of exponentially maturing coupon bonds with a constant amortization rate $\xi>0$. More specifically, each instant there are $\xi F_t dt$ units of required principal repayments from maturing bonds, corresponding to an average bond maturity of $1/\xi$. Debt retirement in this fashion is similar to a sinking fund that continuously buys back debt at par. Together with the aggregate coupon of $cF_t dt$, over $[t,t+dt]$ equity holders are required to pay debt holders the flow payment of $(c+\xi)F_t\,dt$ in order to avoid default.

In the main analysis, we assume investors recover zero value from the assets-in-place when equity holders default. The key implication of this assumption, which will simplify our analysis,

[5] Alternatively, we can interpret the model as written under a fixed risk-neutral measure that is independent of the firm's capital structure decision.

[6] We have simplified notation by assuming the jump size $\zeta(Y_t)$ conditional on cash flow Y_t is deterministic. We can easily generalize the model to allow a random jump size $\tilde{\zeta}(Y_t)$, as long as the law of $\tilde{\zeta}(Y_t)$ depends on Y_t only.

[7] The coupon rate c is exogenously given in our model, so newly issued debt might not be issued at par. In practice, there may be limits/adjustments to the tax deductibility of the coupon if it is far from the par coupon rate. For simplicity, we ignore the tax consequences of non-par debt issuance for this paper.

is that *debt seniority becomes irrelevant*. Because there are no claims to divide in default, old debt holders are *not* directly diluted by new debt holders, where "direct dilution" refers to a decrease in the share of bankruptcy proceeds going to prior creditors.

We make the zero recovery value assumption to emphasize that our results are not driven by the direct dilution that arises when issuing pari-passu debt (e.g., Brunnermeier and Oehmke, 2014; Dangl and Zechner, 2016). Instead, there is an "indirect dilution" effect in our model because the value of existing debt is adversely affected by an increased likelihood of default once new debt is issued. This indirect dilution is a form of debt overhang: shareholders exercise their default option earlier if the firm is more indebted. This indirect dilution effect would arise even if debt were fully prioritized, with new debt always strictly junior to existing debt.[8]

In contrast, in Section 3.4 we will consider the case with a positive recovery value in which the firm can issue new pari passu debt. There we will demonstrate that the threat of direct dilution in fact *reduces* the equilibrium level of debt issued by the firm prior to default.

Equity holders control the outstanding debt F_t through endogenous issuance/repurchase policy $d\Gamma_t$, where Γ_t represents the cumulative debt issuance over time. We focus our main analysis on a class of equilibria in which equity holders find it optimal to adjust the firm's outstanding debt smoothly with order dt. More specifically, we conjecture that at each instant the adjustment to existing debt is $d\Gamma_t = G_t dt$, where G_t specifies the rate of issuance at date *t*, and verify later that other issuance policies, including discrete ones, are suboptimal in equilibrium. From now on, we call this equilibrium a "smooth" equilibrium, and call G_t the equity holders' issuance policy, which could be issuing new debt if $G_t > 0$ or repurchasing existing debt in which case $G_t < 0$. Given our debt maturity assumption, the evolution of outstanding face value of debt F_t is given by

$$dF_t = \left(G_t - \xi F_t\right) dt . \tag{2}$$

[8] Indeed, our qualitative results still hold with a positive recovery rate when new debt must be junior to existing claims. Extending the model in this way adds significant complexity, however, as debt securities issued at different times have distinct prices. In contrast, given zero recovery or pari passu debt, all debt is identical independent of the timing of issuance.

Thus, the face value of debt will grow only if the rate of issuance more than offsets the contractual retirement rate. To highlight the economic forces at play, and in contrast to the bulk of the literature, we assume zero transaction costs in issuing or repurchasing debt.[9]

Given the equity holders' expected issuance/repurchase policy $\{G_t\}$, debt holders price the newly issued or repurchased debt in a competitive market. Denote by p_t the endogenous debt price per unit of promised face value; note that in equilibrium, p_t will reflect creditors' expectations regarding future leverage decisions. Then over $[t, t+dt]$ the net cash flows to equity holders are equal to

$$\left(\underbrace{Y_t}_{\text{operating cash flow}} - \underbrace{\pi(Y_t - cF_t)}_{\text{tax payment}} - \underbrace{(c+\xi)F_t}_{\text{debt interest \& principal}} + \underbrace{G_t p_t}_{\text{debt issuance/repurchase}} \right) dt. \tag{3}$$

The firm continues to operate until the operating cash flow Y_t drops low enough, relative to the outstanding debt level F_t, that equity holders find it optimal to default on their contractual obligation to debtholders. As in the literature (Leland 1994, 1998), shareholders cannot commit to a certain default policy, but instead default strategically. After default, debt holders take over the firm but recover zero by assumption (for the positive recovery case, see Section 3.4).

2.2. Equilibrium Analysis

We focus on Markov perfect equilibria in which the two payoff-relevant state variables are: the firm's *exogenous* operating cash flow Y_t, and the outstanding aggregate debt face value F_t, which is an *endogenous* state variable. We will analyze value function $V(Y_t, F_t)$ for equity and the debt price $p(Y_t, F_t)$. Denote by τ_b the equilibrium default time; that is, the first time that the state (Y_t, F_t) falls into the endogenous default region, which we denote by $\mathcal{B}$. Prior to default, i.e., for $(Y_t, F_t) \notin \mathcal{B}$, given future issuance policies and debt prices $\{(G_s, p_s) : s > t\}$, the market value of equity is equal to

[9] It is common in the dynamic capital structure literature, e.g. Fischer, Heinkel, and Zechner (1989) and Leland, Goldstein, and Ju (2000), to assume that firms--in order to adjust their capital structure--have to buy back all of their existing debt and then reissue new debt, and that there is a positive adjustment cost associated with this transaction. We eliminate this artificial constraint to highlight equity holders' intrinsic incentives to adjust leverage at any time.

$$V(Y,F) = \mathrm{E}_t\left[\int_t^{\tau_b} e^{-r(s-t)}\left[Y_s - \pi(Y_s - cF_s) - (c+\xi)F_s + G_s p_s\right]ds \,\middle|\, Y_t = Y, F_t = F\right]. \quad (4)$$

Because debtholders receive both coupon and principal payments until the firm defaults, and the firm recovery upon default is assumed to be zero, the market price of debt is given by

$$p(Y,F) = \mathrm{E}_t\left[\int_t^{\tau_b} e^{-(r+\xi)(s-t)}(c+\xi)dt \,\middle|\, Y_t = Y, F_t = F\right]. \quad (5)$$

An Optimality Condition

Recall that we are interested in an equilibrium when there is no commitment by equity holders to future leverage policies. Thus, at any point in time, the issuance policy G_s for $s > t$ has to be optimal in solving the equity holders' instantaneous maximization problem at time s, given equity's value function and equilibrium debt prices.

In this section we consider the necessary and sufficient conditions for the optimality of the debt issuance policy G_t. The Hamilton-Jacobi-Bellman (HJB) equation for equity holders is

$$\begin{aligned} \underbrace{rV(Y,F)}_{\text{required return}} = \max_G &\left[\underbrace{Y - \pi(Y - cF)}_{\text{after-tax cash flow}} - \underbrace{(c+\xi)F}_{\text{coupon\&principal payment}} + \underbrace{Gp(Y,F)}_{\text{new debt issuance}} + \underbrace{(G - \xi F)V_F(Y,F)}_{\text{evolution of } dF}\right] \\ &+ \underbrace{\mu(Y)V_Y(Y,F) + \frac{1}{2}\sigma(Y)^2 V_{YY}(Y,F) + \lambda(Y)\left[V(Y+\zeta(Y)) - V(Y)\right]}_{\text{evolution of } dY} \end{aligned} \quad (6)$$

In the first line, the objective is linear in G with a coefficient of $p(Y,F) + V_F(Y,F)$, which represents the (endogenous) marginal benefit of the revenue from a debt sale net of the marginal cost of the future debt burden on shareholders. If equity holders find it optimal to adjust debt smoothly, then it must be that this coefficient equals zero, i.e.

$$p(Y,F) + V_F(Y,F) = 0. \quad (7)$$

This first-order condition (FOC) must hold for any (Y,F) along the equilibrium path.[10] But to be sure that this policy is globally optimal, we must verify that there is no discrete adjustment to the debt level that shareholders would prefer. We show next that global optimality holds if and only if the debt price is weakly decreasing in the firm's total debt, i.e., $p_F(Y,F) \le 0$.

PROPOSITION 1 (OPTIMALITY OF SMOOTH DEBT POLICY). Suppose the debt price $p = -V_F(Y,F)$ is weakly decreasing in the total face value F of the firm's debt, i.e., the value of equity $V(Y,F)$ is convex in F. Then condition (7) implies that the policy G_t is an optimal debt issuance policy for shareholders. Conversely, if the policy G_t is optimal, then $p = -V_F(Y,F)$ is weakly decreasing in F for all equilibrium states (Y,F).

PROOF. Equity holders are solving the following problem each moment along the optimal path

$$\max_\Delta V(Y,F+\Delta) + \Delta \cdot p(Y,F+\Delta) - V(Y,F). \tag{8}$$

For the proposed smooth policy to be optimal, $\Delta = 0$ must be optimal in (8). This problem has the first-order condition $V_F + p + \Delta\, p_F = 0$ at $\Delta = 0$, which implies that $p = -V_F$. To check for global optimality, suppose that equity holders choose any $\Delta > 0$. Then equity's gain is

$$\begin{aligned} V(Y,F+\Delta) - V(Y,F) + \Delta \cdot p(Y,F+\Delta) &= \int_0^\Delta V_F(Y,F+\delta)\,d\delta + \int_0^\Delta p(Y,F+\Delta)\,d\delta \\ &\le \int_0^\Delta V_F(Y,F+\delta)\,d\delta + \int_0^\Delta \underbrace{p(Y,F+\delta)}_{\text{monotonicity of } p}\,d\delta \\ &= \int_0^\Delta \underbrace{\left[V_F(Y,F+\delta) + p(Y,F+\delta)\right]}_{\text{FOC in Equation (7)}}\,d\delta = 0 \end{aligned} \tag{9}$$

where in the second line we have used the condition that p is weakly monotone in F. Note the above inequalities still hold if $\Delta < 0$; in this case $p(F+\Delta) \ge p(F+\delta)$ but $d\delta < 0$. Finally, the

[10] This relation holds trivially in the default region $\mathscr{B}$, as for defaulted firm the debt price $p = 0$ and $V(Y,F) = 0$ implies $V_F(Y,F) = 0$ as well. It is worth pointing out that the zero-bankruptcy-recovery assumption is not necessary for $p = 0$ at default. Even with a strictly positive recovery, as long as newly issued debt is junior to any pre-existing debt, newly issued debt is worthless at the moment of default as existing debt gets all the recovery. (Alternatively, if the firm can issue pari passu debt, then right before default it will issue new debt and dilute existing creditors, so much so that the recovery value of all existing debt converges to zero, as we show in Section 3.4.)

condition that the debt price is weakly decreasing in F implies that equity's value is convex in F, i.e. $p_F \le 0 \;\Leftrightarrow\; V_{FF} \ge 0$.

Now we prove the second part. First of all, it is easy to see $p(Y,F)$ is continuous in F from (5). If $p(Y,F)$ is weakly increasing in F, the above argument in (9) implies that any Δ is a profitable deviation. But if $p(Y,F)$ is not monotone in F, then for some Y, due to continuity there must exist two face values F_1 and F_2 with $F_1 < F_2$, so that $p(Y,\hat{F}) < p(Y,F_1) = p(Y,F_2)$ for $\hat{F} \in [F_1,F_2]$. Now setting $\Delta = F_2 - F_1 > 0$ and $F = F_1$ in (9) leads to a strictly positive deviation gain. ■

Equity Valuation

The First-Order Condition (FOC) in (7), which implies a zero-profit condition for equity holders in adjusting the debt burden instantaneously, has deep implications for the equilibrium in our model. Plugging condition (7) into the equity HJB equation (6), we can see that the sum of terms involving G equals zero, leading to the following revised HJB equation for equity:

$$\begin{aligned} rV(Y,F) &= Y - \pi(Y - cF) - (c+\xi)F + \mu(Y)V_Y(Y,F) - \xi F V_F(Y,F) \\ &\quad + \tfrac{1}{2}\sigma(Y)^2 V_{YY}(Y,F) + \lambda(Y)\left[V\left(Y+\zeta(Y),F\right) - V(Y,F)\right]. \end{aligned} \tag{10}$$

This equation says that in the no-commitment equilibrium, the equity value can be solved as if there is no debt adjustment $G_t = 0$, except for the natural retirement at rate ξ.

Intuitively, because equity holders gain no marginal surplus from adjustments to debt, their equilibrium payoff must be the same as if they never issue/repurchase any debt. The important implication of this observation is that we can solve for the equilibrium equity value $V(Y,F)$, even without the knowledge of the equilibrium debt price – $p(Y,F)$ does not enter equation (10)). Therefore, we have the following key result:

PROPOSITION 2 (NO-TRADE EQUITY VALUATION). Let $V^0(Y,F)$ be the value of equity that solves (10) in which the firm were committed not to issue or buyback debt ($G_t = 0$). Then the value V of equity in any smooth equilibrium is equal to V^0.

PROOF. Immediate from (10) and the fact that the boundary conditions are unchanged. ■

This result, while perhaps striking at first, is analogous to the Coase (1972) conjecture for durable goods monopoly: the firm is a monopolist issuer of its own debt. When the firm is unable to commit to restricting its future sales, it cannot resist the temptation to trade aggressively, so much so that any surplus from trading gets dissipated in equilibrium.[11]

Optimal Debt Issuance

Given the equity value V, we can invoke the FOC in (7) to obtain the equilibrium debt price $p(Y,F) = -V_F(Y,F)$. Finally, to confirm that this outcome indeed represents an equilibrium, we must verify whether $p(Y,F) = -V_F(Y,F)$ is weakly decreasing in F, or equivalently that the equity value is convex in F.

Now we solve for the optimal leverage policy *G*. We have determined the equilibrium debt price using the optimality $p = -V_F$, where the equity value *V* is equivalent to the no-trade value from (10) and therefore independent of *G*. On the other hand, we can also calculate the debt price directly based on debt holders expected cash flows using (5). This calculation will depend on the timing of default, which does depend on the rate of debt issuance. Thus the equilibrium leverage policy *G* must be such that these two methods of valuing the debt are consistent with each other.[12]

The next two steps follow the idea outlined above. First, let us consider the HJB equation that should hold for the debt price from (5), which is given by

$$\begin{aligned}\underbrace{rp(Y,F)}_{\text{required return}} = \underbrace{c}_{\text{coupon}} + \underbrace{\xi\left(1-p(Y,F)\right)}_{\text{debt retirement}} + \underbrace{(G^* - \xi F)\,p_F(Y,F)}_{\text{evolution of debt } dF} + \\ \underbrace{\mu(Y)\,p_Y(Y,F) + \tfrac{1}{2}\sigma(Y)^2\,p_{YY}(Y,F) + \lambda(Y)\left[p\left(Y+\zeta(Y),F\right) - p(Y,F)\right]}_{\text{evolution of cash flow } dY}.\end{aligned} \tag{11}$$

Next, starting with the HJB equation (10) for $V(Y,F)$, if we differentiate by F and use the optimality condition $p = -V_F$, we obtain

[11] A closely related result appears in DeMarzo and Urosevic (2001) in a model of trade by a large shareholder trading off diversification benefits and price impact due to reduced incentives. In equilibrium, share prices are identical to those implied by a model with no trade. Similarly, the monopolist buyer in Daley and Green (2016) who cannot commit to his/her future strategy gains nothing from the ability of screening (sellers with different types).

[12] Intuitively, if $G = 0$ and the firm never issues additional debt, the debt price would exceed its marginal cost to shareholders, $-V_F$, due to the incremental tax shield. By increasing the rate of issuance, the likelihood of default will increase and the price of debt will fall to the point that (7) holds.

$$-rp(Y,F) = \pi'(Y-cF)c - (c+\xi) + \xi\, p(Y,F) + \xi F p_F(Y,F)$$
$$-\mu(Y)\, p_Y(Y,F) - \tfrac{1}{2}\sigma(Y)^2 p_{YY}(Y,F) + \lambda(Y)\big[-p\big(Y+\zeta(Y),F\big) + p(Y,F)\big]. \tag{12}$$

Although equation (12) is written in terms of the debt price p, we emphasize that it follows mechanically from the valuation equation (10) for equity, together with the FOC (7) for the optimal issuance policy. Finally, adding (12) to (11), we obtain a simple expression for G shown below:

> **PROPOSITION 3 (EQUILIBRIUM DEBT ISSUANCE).** Let $V(Y,F)$ be the no-trade value of equity. If V is convex in F, then there exists a unique smooth equilibrium with debt issuance policy
>
> $$G^*(Y,F) = \frac{\pi'(Y-cF)c}{-p_F(Y,F)} = \frac{\pi'(Y-cF)c}{V_{FF}(Y,F)}. \tag{13}$$
>
> Under this policy, the debt price given by (5) satisfies $p = -V_F$.

PROOF. For a smooth policy to be optimal, (7) is necessary. But then (6) and (7) imply (12), which combined with (11) imply (13). Then $p = -V_F$ follows since their HJB equations and boundary conditions are equivalent, and the global optimality of the policy (13) follows from Proposition 1 and convexity of *V*. ■

Note that the convexity of equity value *V* implies that, no matter how high the current level of debt, the rate of issuance G^* is always positive provided a strictly positive tax benefit $\pi' > 0$. We can interpret the policy (13) as follows. The rate of issuance of debt is such that the rate of devaluation of the debt induced by new issuances just offsets the marginal tax benefit associated with the coupon payments:

$$G^*(Y,F)\; p_F(Y,F) = \pi'(Y-cF)\, c\ . \tag{14}$$

As shown by (14), if there were no tax subsidy ($\pi' = 0$), then $G^*(Y,F) = 0$, that is, without a tax subsidy equity holders choose not to increase debt. On the other hand, they choose not to actively reduce debt via buybacks either, despite that fact that there are deadweight costs of bankruptcy. This result is consistent with the leverage ratchet effect of Admati et al. (2017) – even if the firm's current leverage is excessive, equity holders never actively reduce debt but always have an incentive to increase debt when it provides a marginal tax benefit.

Discrete Optimization

Note that in equilibrium, because $p = -V_F$, the value of equity is the same for any smooth issuance policy G, yet the optimal policy G^* is uniquely determined such that shareholders remain indifferent. In a sense, our characterization of G^* is analogous to that of a mixed strategy equilibrium in which each player is indifferent to her choice of action, yet her equilibrium strategy is uniquely determined to keep the other player indifferent.

Shareholder indifference regarding the issuance policy is, however, an artifact of the continuous-time limit. If we were to compute the equilibrium as the limit of a discrete-time model, the optimal policy G^* would arise as the result of a strict optimization by shareholders. To see this result heuristically, suppose that the firm issues debt Δ which is fixed over the next dt instant, and let p and V be the end-of-period debt price and equity value functions, respectively. The firm would then pay additional interest of $\Delta c dt$, and thus its earnings would decline by $\left(1-\pi'\left(Y-cF\right)\right)\Delta c dt$ on an after-tax basis.[13] Because the bonds trade for a cum-coupon price of $cdt + p(Y, F+\Delta)$, shareholders would choose Δ to solve:[14]

$$\max_{\Delta} \underbrace{-\left(1-\pi'\left(Y-cF\right)\right)\Delta c dt}_{\text{After-tax interest payment}} + \underbrace{\Delta\left[cdt + p(Y, F+\Delta)\right]}_{\text{Debt proceeds}} + V(Y, F+\Delta) \tag{15}$$

Equation (15) has the first-order condition

$$\Delta = \frac{\pi'\left(Y-cF\right)cdt + \overbrace{\left(p+V_F\right)}^{\text{zero by equilibrium condition (7)}}}{-p_F} = \frac{\pi'\left(Y-cF\right)c}{-p_F}dt, \tag{16}$$

which exactly coincides with (13). Hence, we can interpret G^* as the strictly optimal issuance rate when the firm has "infinitesimal" commitment power over $\left[t, t+dt\right]$ in a discrete-time setting.

[13] Here we are ignoring terms of order dt^2 or higher which would arise if the marginal tax rate is not locally constant.
[14] Recall that p is the end-of-period bond price. If sold earlier it will trade for a higher price that includes the initial coupons. Also, we assume the new debt issuance occurs after the current period's default decision and principal repayments; changing the timing would introduce terms of order dt^2 without altering the conclusion.

Summary

In sum, for the general model in which equity holders are free to issue or repurchase any amount of debt at the prevailing market price, one can solve for the no commitment equilibrium as follows:

(i) Use (10) to solve for the equity holder's value function $V(Y,F)$ by setting $G=0$, i.e. as if equity holders commit to not issue any future debt;

(ii) Set the debt price $p(Y,F)=-V_F(Y,F)$;

(iii) Check the global optimality condition by verifying the debt price $p(Y,F)$ is weakly decreasing in aggregate debt F, or equivalently $V(Y,F)$ is convex in F;

(iv) Finally, given $p(Y,F)$ we can solve for the optimal time consistent issuance policy $G^*(Y,F)$ from (13).

In the remainder of the paper we will use this methodology to analyze several standard settings.

3. A Closed-Form Solution

We now apply the general methodology developed in the previous section to the widely used framework of a lognormal cash flow process.[15] The results from Section 2 allow us to fully characterize an equilibrium in closed form, and evaluate the corresponding leverage dynamics. We also extend the model to allow for jumps to cash flows, and show that the solution is qualitatively unchanged. Finally, Section 3.4 studies the case of a positive recovery value; somewhat surprisingly, because equity holders are able to dilute existing creditors, a positive recovery value makes the debt price more sensitive to new issuance, thereby reducing the equilibrium level of debt.

3.1. Log-Normal Cash Flows

In the special case of lognormal operating cash flow, Y_t follows a geometric Brownian motion:

[15] This setting is consistent with e.g. Merton (1974), Fischer et al. (1989), Leland (1994), Leland and Toft (1996), and follows the development of starting from cash flows rather than firm value as in Goldstein et al. (2001).

$$\mu(Y_t) = \mu Y_t \text{ and } \sigma(Y_t) = \sigma Y_t \text{, with } r > \mu . \tag{17}$$

Given the scale invariance of the firm in this setting, we analyze the model using a unidimensional state variable equal to operating cash flow scaled by the outstanding face value of debt,

$$y_t \equiv Y_t / F_t . \tag{18}$$

As an interpretation, y_t is proportional to the firm's *interest coverage ratio* y_t / c, that is, the ratio of operating income Y_t to total interest expense cF_t, a widely used measure of leverage and financial soundness. Alternatively, $1/ y_t$ expresses the amount of debt as a multiple of the firm's cash flow (EBIT).

To maintain homogeneity, we assume a constant tax rate so that

$$\pi(Y_t - cF_t) = \overline{\pi} \cdot (y_t - c) \cdot F_t , \tag{19}$$

where the positive constant $\overline{\pi} > 0$ is the marginal corporate tax rate that applies to both losses and gains.[16] With this setting, we conjecture and verify that the equity value function $V(Y,F)$ and debt price $p(Y,F)$ are homogeneous so that

$$V(Y,F) = V\left(\frac{Y}{F},1\right)F \equiv v(y)F \text{ and } p(Y,F) = p\left(\frac{Y}{F},1\right) \equiv p(y). \tag{20}$$

We will solve for the (scaled) equity value function $v(y)$ and debt price $p(y)$ in closed form.

Given the evolution of our state variables Y_t and F_t:

$$dY_t = \mu Y_t dt + \sigma Y_t dZ_t , \text{ and } dF_t = (G_t - \xi F_t) dt , \tag{21}$$

the scaled cash-flows evolve as

$$\frac{dy_t}{y_t} = (\mu + \xi - g_t) dt + \sigma dZ_t , \text{ where } g_t \equiv G_t / F_t . \tag{22}$$

[16] The methods developed here could also be applied with different marginal tax rates for losses versus gains, e.g. $\pi(y - c) = \overline{\pi} \cdot \max(y - c, 0)$, though we do not pursue that here.

As (22) shows, because the debt F_t grows in a locally deterministic way, the scaled cash flows grow with the same volatility as total cash flows. Their expected growth rate, however, is reduced by the net growth rate of the debt $g_t - \xi$, where ξ is the debt amortization rate and $g_t = G_t / F_t$ is the endogenous debt issuance rate. The more debt the firm issues, the faster the scaled cash flow declines.

When the scaled cash flow y_t falls below some endogenous default boundary y_b, equity holders are no longer willing to service the debt, and therefore choose to strategically default. At that event, equity holders walk away and debt holders recover nothing by the assumption of a zero liquidation value.

3.2. Model Solution

Recall from Section 2 that we can solve for the equilibrium equity value *as if* $g_t = 0$ and equity holders do not actively adjust the firm's debt, even though they will do so in equilibrium. Using the fact that

$$V_Y\left(Y,F\right) = v'\left(y\right),\ V_F\left(Y,F\right) = v\left(y\right) - yv'\left(y\right), \text{ and } FV_{YY} = v''\left(y\right), \tag{23}$$

we can rewrite (10) with lognormal cash flows in terms of scaled cash-flow y as follows:

$$\left(r+\xi\right)v\left(y\right) = (y-c-\xi) - \overline{\pi}(y-c) + \left(\mu+\xi\right)yv'\left(y\right) + \frac{1}{2}\sigma^2 y^2 v''\left(y\right). \tag{24}$$

There are two boundary conditions for the ODE (24). When $y = y_b$, equity is worthless so $v\left(y_b\right) = 0$. On the other hand, when $y \to \infty$, default becomes unimportant and we can treat the debt as riskless, and hence the equity value converges to

$$\overline{v}(y) \equiv \underbrace{\frac{y(1-\overline{\pi})}{r-\mu}}_{\text{unlevered asset value}} + \underbrace{\frac{\overline{\pi}c}{r+\xi}}_{\text{tax shield}} - \underbrace{\frac{c+\xi}{r+\xi}}_{\text{bond value}} \equiv \phi y - \rho\ , \tag{25}$$

where $\phi \equiv \frac{1-\overline{\pi}}{r-\mu}$ is the unlevered valuation multiple for the firm, and $\rho \equiv \frac{c(1-\overline{\pi})+\xi}{r+\xi}$ is the after-tax cost to the firm of a riskless bond. Finally, the default boundary y_b is determined by the smooth-pasting condition $v'(y_b)=0$. Solving (24) with these boundary conditions, we obtain the following characterization of the equity value.

PROPOSITION 4. Given a constant tax rate $\bar{\pi}$, and letting

$$\gamma \equiv \frac{\left(\mu+\xi-0.5\sigma^2\right)+\sqrt{\left(\mu+\xi-0.5\sigma^2\right)^2+2\sigma^2\left(r+\xi\right)}}{\sigma^2}>0, \tag{26}$$

the equity value function and optimal default boundary are given by

$$v(y)=\phi y-\rho\left(1-\frac{1}{1+\gamma}\left(\frac{y}{y_b}\right)^{-\gamma}\right) \text{ and } y_b=\frac{\gamma}{1+\gamma}\frac{\rho}{\phi}. \tag{27}$$

PROOF. See the Appendix. ■

Having solved for the value of equity, recall from (7) that we can determine the equilibrium debt price from the FOC $p(y)=-V_F(Y,F)$. Then from (23), and using (27), we have

$$p(y)=-V_F=yv'(y)-v(y)=\rho\left(1-\left(\frac{y}{y_b}\right)^{-\gamma}\right). \tag{28}$$

Recall we need to verify the optimality of the issuance policy by checking the monotonicity of the equilibrium debt pricing function. It is easy to see that $p'(y)>0$ in (28), i.e. the greater the scaled cash flow the higher the debt price. As a result, the key condition in **PROPOSITION 1** – that the debt price decreases with total debt – follows because

$$p_F(Y,F)=p'(y)\cdot\left(-\frac{Y}{F^2}\right)=-\frac{y^2v''(y)}{F}<0. \tag{29}$$

Finally, we can apply **PROPOSITION 3** combined with (29) to derive the equilibrium debt issuance policy:

PROPOSITION 5. Given a constant tax rate $\bar{\pi}$, the equilibrium issuance policy is

$$g^*(y)=\frac{G^*}{F}=\frac{\bar{\pi}c}{-Fp_F(Y,F)}=\frac{\bar{\pi}c}{yp'(y)}=\frac{\bar{\pi}c}{y^2v''(y)}=\frac{\bar{\pi}c}{\rho\gamma}\left(\frac{y}{y_b}\right)^{\gamma}. \tag{30}$$

New debt issuance $g^*(y)$ is always positive, and is increasing in the scaled cash flow y.

Note that g^* represents the issuance rate as a proportion of the current debt level F; that is, total issuance is $G^* = Fg^*(y)$. And although $g^*(y)$ approaches infinity as $F \to 0$, in Section 4.1 we will derive the debt dynamics explicitly starting from $F = 0$ and show that under the optimal policy the firm's outstanding debt follows a continuous sample path with no jumps.

Thus, with lognormal cash flows, we can fully characterize equilibrium debt dynamics and security pricing in closed form. Based on the equilibrium values for both equity and debt, total firm value (or total enterprise value, TEV) can be expressed as a multiple of the firm's cash flow (i.e. TEV to EBIT) as

$$\frac{v(y)+p(y)}{y} = v'(y) = \phi - \frac{1}{y_b}\frac{\rho\gamma}{1+\gamma}\left(\frac{y}{y_b}\right)^{-\gamma-1} = \phi\left[1-\left(\frac{y}{y_b}\right)^{-\gamma-1}\right], \tag{31}$$

where the first equality follows from the equilibrium condition for the debt price, and the last equality uses the expression of y_b in **PROPOSITION 4**.

Note that the firm's TEV multiple is strictly increasing with the scaled cash flow y. Consequently, holding the level of cash flows fixed, total firm value *decreases* with leverage. Although there are tax benefits associated with debt, the firm issues debt sufficiently aggressively that the cost of debt rises to offset the tax benefits. An immediate implication of this result is that starting with zero leverage, there is no incentive for the firm to increase leverage discretely. Instead, in equilibrium, the firm will issue debt gradually according to (30). This behavior is in stark contrast to models with commitment, which we discuss further in Section 5.

3.3. Upward Jumps

In the no commitment equilibrium, the firm's debt level evolves continuously according to equation (30). This smooth issuance policy might be thought to depend on continuity of cash flows and asset values in the diffusion setting. In this section we extend our model to allow the firm's cash flows to jump discontinuously, for example in response to new product development, and show that our prior solution, in which shareholders issue debt smoothly, is essentially unchanged.

Consider a jump-diffusion model in which cash flows occasionally jump from Y_t to θY_t for some constant $\theta > 1$. Specifically,

$$dY_t = \hat{\mu} Y_t dt + \sigma Y_t dZ_t + (\theta - 1) Y_{t-} dN_t , \tag{32}$$

where dN_t is a Poisson process with constant intensity $\lambda > 0$.[17] In this extension, due to upward jumps, the effective expected asset growth rate becomes

$$\mu \equiv \hat{\mu} + \lambda(\theta - 1), \tag{33}$$

and we continue to assume $\mu < r$ to ensure that the unlevered firm value is bounded.

As before, we can solve for the equity value as if shareholders commit not to issue any new debt. Because (32) still maintains scale-invariance, $V(Y, F) = F \cdot v(y)$ continues to hold, and the HJB equation for the equity value becomes

$$(r + \xi) v(y) = (1 - \overline{\pi})(y - c) - \xi + (\hat{\mu} + \xi) y v'(y) + \frac{1}{2}\sigma^2 y^2 v''(y) + \lambda\big(v(\theta y) - v(y)\big). \tag{34}$$

The last term in equation (34) captures upward jumps. The usual boundary conditions apply: When $y \to \infty$ so leverage is negligible, default risk disappears and $v(y) \to \overline{v}(y)$; while at the point of default, we have value-matching $v(y_b) = 0$ and smooth-pasting $v'(y_b) = 0$.

Somewhat surprisingly, even with jumps, equilibrium security prices and debt dynamics have exactly the same form we derived in the diffusion-only case:

PROPOSITION 6. Suppose cash flows evolve as a log-normal diffusion with upward jumps as in (32). Then the equilibrium equity and debt values, and debt issuance policy are given by (27), (28), and (30) respectively, with γ replaced by $\hat{\gamma}$, which is the unique positive root of

$$W(\hat{\gamma}) = \lambda \theta^{-\hat{\gamma}} + \frac{\sigma^2}{2}\hat{\gamma}^2 - \left(\hat{\mu} + \xi - \frac{\sigma^2}{2}\right)\hat{\gamma} - (r + \xi + \lambda) = 0 . \tag{35}$$

[17] While we focus on upward jumps with a fixed size, allowing the upward jump to be stochastic is straightforward. Downward jumps introduce an extra complication due to jump-triggered default, in addition to diffusion-triggered default. See footnote 19 for more details.

PROOF. See the Appendix. ■

Consequently, although the firm's profitability (i.e., cash-flow Y_t) may jump up discretely, the equilibrium debt issuance policy continues to be smooth in the sense that it remains of order dt. In response to positive jumps in the firm's profitability, shareholders increase the speed of debt issuance, but do not issue a discrete amount of debt immediately. Consequently, leverage falls discretely before gradually mean-reverting. This property holds even we set $\sigma^2 = 0$ so that the firm's cash flows only grow with discrete jumps.[18,19]

3.4. Positive Recovery Value

Thus far we have assumed that in the event of default the liquidation value of the firm is zero. Under this assumption, there is no difference between junior or senior debt, which rules out any direct dilution motive for issuing debt.

What if the firm has a positive recovery value in default, and the firm can issue pari passu or even senior debt, so that there is a dilution motive for debt issuance? Specifically, suppose that given cash flows Y, the firm has a liquidation value $L(Y)$ equal to a fraction of its unlevered value:

$$L(Y) \equiv \alpha\phi Y \text{ for } \alpha \in [0,1). \tag{36}$$

These liquidation proceeds are paid to the firm's creditors. But if the firm can issue senior or pari passu debt without restriction, then by issuing new debt shareholders can dilute the claim of existing creditors in default. Indeed, at the moment of default, shareholders have an incentive to issue new debt to dilute the existing creditors fully, so that as a result existing creditors earn a zero recovery. Because shareholders receive the proceeds from the new debt issued, this scenario is equivalent to equity holders having the option to default on existing creditors and recover $L(Y)$.[20]

---

[18] When the diffusion term vanishes, we must impose $\mu + \xi < 0$ so that cash flows decline faster than debt matures between jumps. Otherwise, it is optimal for the firm to sustain 100% debt financing without risking default.

[19] If we allow for negative jumps, there is an additional complication that jumps may trigger default. Nonetheless, the analysis in Chen and Kou (2009), with certain special assumptions on jump distributions, suggests that one can still solve for the equity valuation in closed-form. As long as the equity value function remains convex, the key qualitative property of smooth debt issuance policy continues to hold in general jump-diffusion models.

[20] In other words, it is as if there is a complete violation of absolute priority so that equity holders receive the entire recovery value of the firm (while debt holders recover nothing).

Interestingly, we can show that in this case the resulting equilibrium is equivalent to one in which only a fraction $(1-\alpha)$ of the firm's cash flows can be pledged to creditors (with shareholders owning the remaining α fraction of non-pledgeable cash flows separately). The optimal default and issuance policies derived above continue to apply with the relevant measure of cash flow equal to just the pledgeable component $(1-\alpha)y$. The value of equity is then the sum of the value from the pledgeable and non-pledgeable parts. The following proposition formalizes this result:

PROPOSITION 7. Suppose cash flows evolve as a log-normal diffusion as in (17), all debt is pari passu, and in the event of default the firm is worth $L(Y)$ as in (36). Then the equilibrium equity and debt values, default boundary, and debt issuance policy are:

$$v^L(y) = v(y(1-\alpha)) + L(y),\ y_b^L = \frac{y_b}{1-\alpha}, \tag{37}$$

$$p^L(y) = p(y(1-\alpha)), \text{ and } g^L(y) = g^*(y(1-\alpha)). \tag{38}$$

PROOF. See the Appendix. ■

In this scenario, the smooth issuance policy g^L applies only up to the default boundary y_b^L. At the moment of default, the firm issues an "infinite" amount of debt to dilute fully its existing creditors. This characterization is of course extreme – in practice we would expect to see dilution occur leading up to default – but nonetheless captures the idea that if creditors are not protected by prioritization rules their expected recovery value will be nil. The main qualitative effect of a positive recovery rate is instead to raise the value of equity (i.e. $v^L(y) > v(y)$) and *reduce* the equilibrium level of debt prior to default (because debt issuance will match that of a firm with proportionally lower cash flows).

It is interesting to observe that $\partial g^L / \partial y > 0$, so that shareholders issue less debt when the firm edges closer to default (but before actual default). Even more surprising, we have $\partial g^L / \partial \alpha < 0$ so that a firm with a dilution motive issues less debt prior to default relative to the baseline case without a dilution motive. The extra dilution motive is instead reflected by the more aggressive default policy (and the associated infinite dilution at default), whereas the reduced pledgeability of cash flows lowers debt capacity prior to default. This result squares nicely with Dangl and

Zechner (2016) who consider similar pari passu debt and positive recovery, but shareholders are constrained by an upper bound on the rate of debt issuance. Because of this constraint, shareholders in Dangl and Zechner (2016) raise debt at the maximum speed possible for some period prior to default.

4. Debt Dynamics

Now that we have solved for the equilibrium debt issuance policy and security pricing, we can analyze the implications for observed debt dynamics. Although lack of commitment leads the firm to always have a positive rate of debt issuance, the countervailing effect of debt maturity and asset growth cause leverage to mean-revert towards a target. We begin by characterizing this target as well as the speed of adjustment. We then consider the implications of alternative debt maturities, and find that in our model shareholders are indifferent to any maturity structure for future debt issuance.

4.1. Target Leverage and Adjustment Speed

From **Proposition 5**, we see that the firm will issue debt at a faster rate when cash flows are high, and the rate of issuance slows as the firm approaches default. Because the mapping from the cash flow to leverage is strictly monotonic, there is a unique level of the scaled cash flow $\hat{y}_{\bar{g}}$ such that new net debt issuances occur at any given rate $\bar{g}$. We can compute $\hat{y}_{\bar{g}}$ from (30) as follows:

$$\hat{y}_{\bar{g}} \equiv y_b \left(\frac{\rho\gamma}{\bar{\pi}c} \cdot \bar{g} \right)^{1/\gamma}. \tag{39}$$

Then $\hat{y}_{\xi}$ (i.e., set $\bar{g} = \xi$) is the target level of the scaled cash flow at which the new issuance exactly balances with retiring of existing debt, and the firm is neither accumulating nor retiring debt.

Figure 2 illustrates the net debt issuance rate given different debt maturities and asset volatilities as a function of the firm's current leverage. This issuance policy causes leverage to mean revert towards a target level $\hat{y}_{\xi}$ at which new issuance just balances debt maturity. Shorter debt maturity increases the speed of mean reversion, whereas lower volatility raises the target level of leverage.

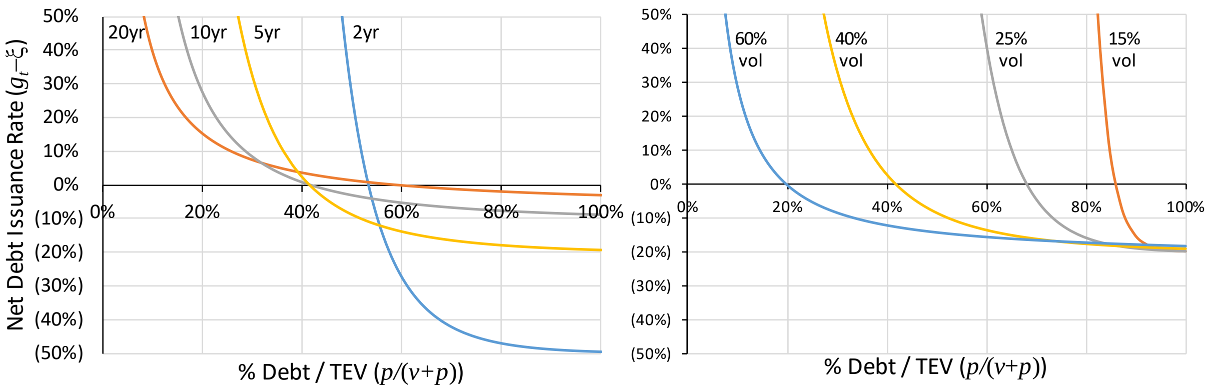

Figure 2: Net Debt Issuance versus Firm Leverage for Different Maturities and Volatilities

Baseline Parameters: $\mu = 2\%,\ \sigma = 40\%,\ \overline{\pi} = 30\%,\ c(1-\overline{\pi}) = r = 5\%, \xi = 20\%$

Without commitment, the firm's debt is path dependent, with the current level of debt equal to the cumulative past issuance net of debt retirement. Because the issuance rate varies with the level of cash flows, this path dependence can be quite complex. Surprisingly, using our expression for g^* in **Proposition 5**, we can derive the evolution of the firm's debt explicitly as a function of the firm's initial debt position and its earnings history, as shown next.

Proposition 8. Given the debt issuance policy g^* and initial debt face value $F_0 \geq 0$, the firm's debt on date t given the cash-flow history $\{Y_s : 0 < s < t\}$ is

$$F_t = \left[F_0{}^{\gamma} e^{-\gamma\xi t} + \int_0^t \gamma\xi \left(\frac{Y_s}{\hat{y}_\xi} \right)^{\gamma} e^{\gamma\xi(s-t)} ds \right]^{1/\gamma} \tag{40}$$

Proof. See the Appendix. ■

Equation (40) implies that the firm's debt today equals an appropriately "discounted average" of the initial debt and a target multiple of the firm's intervening cash flows. This point becomes transparent in the special case of $F_0 = 0$, i.e., when the firm starts with no debt. Setting $F_0 = 0$ in (40), we have

$$F_t = \frac{1}{\hat{y}_\xi} \left(\gamma\xi \int_0^t e^{\gamma\xi(s-t)}\, Y_s^{\gamma} ds \right)^{1/\gamma} \tag{41}$$

Since $\gamma\xi\int_0^t e^{\gamma\xi(s-t)}\,ds = 1-e^{-\gamma\xi t}$ is approximately $\gamma\xi t$ for small t, (41) implies that in equilibrium debt starts at 0 and grows gradually with order of $t^{1/\gamma}$,[21] with the long run debt level dependent on the weighted average of the firm's historical earnings. The weight put on recent cash flows relative to more distant ones is an increasing function of the product $\gamma\xi$. Table 1 shows the weight put on the firm's last 12 months of earnings in the determination of its current debt level. Intuitively, shorter debt maturity (a greater ξ) implies faster repayment of debt principal, allowing leverage to shrink more quickly in the face of declining cash flows. From (26) one can show that higher γ is associated with lower volatility, which makes the firm more aggressive in adding leverage in response to positive cash flows news. Finally, note that the only impact of the tax rate $\overline{\pi}$ is to rescale the debt level through $\hat{y}_{\xi}$.[22]

Weight on Prior Year Earnings in Current Debt Level				
	Volatility			
Maturity	35%	40%	45%	50%
20yr	7%	5%	4%	3%
10yr	19%	15%	12%	10%
5yr	52%	43%	36%	31%
3yr	86%	78%	69%	62%

Table 1: Weight of Prior Year Earnings in Current Debt Level ($1-e^{-\gamma\xi}$). The debt level responds more quickly to current earnings when debt maturity is short and volatility is low.

Equation (41) demonstrates clearly that once the firm is free to adjust leverage over time, equilibrium debt dynamics depart strongly from the standard predictions of tradeoff theory. Figure 3 simulates the evolution of debt for an initially unlevered, issuing debt with a five-year maturity, using the same parameters as Figure 2. While the firm does not issue a large amount of debt immediately, it does increase debt quickly until it approaches approximately 40% of firm value. Once it exceeds that level, the firm issues new debt at a slower rate than its existing debt matures, and the total amount of debt declines. Overall, the firm's debt level evolves gradually

[21] This continuous debt increment at $t=0$ is consistent with the infinite growth rate of $g^*(y)$ as $y\to\infty$.

[22] Note that our model adopts an idealized version of the tax code. In practice, the debt tax shield is not strictly tied to the coupon rate, but includes an adjustment for any discount or premium at the time of issuance. As a result, when leverage and credit risk are high, a firm issuing debt at a discount would enjoy somewhat higher tax shields than when leverage or credit risk are low. For moderate levels of leverage, this would tend to reduce the speed of mean reversion. At very high leverage levels, however, when the firm has operating losses, tax shields must be deferred and the rate of debt issuance would slow at a faster rate than in our model.

based on a weighted average of past earnings. This gradual adjustment of debt towards a target level, as simulated in Figure 3, resembles the evolution of debt observed in practice (see, for example, Figure 1).

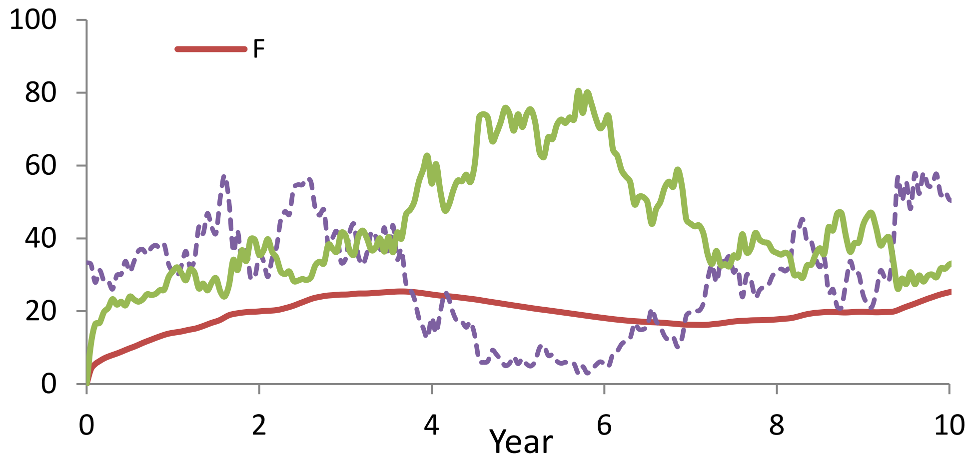

Figure 3: Simulation of Debt Evolution

($\mu = 2\%, \sigma = 40\%, \overline{\pi} = 30\%, c(1-\overline{\pi}) = r = 5\%, \xi = 20\%$)

4.2. Debt Maturity Structure

Our model considers a constant debt maturity structure in which all debt has an expected maturity of $1/\xi$. This assumption is common in much of the dynamic capital structure literature which treats the debt maturity structure as a parameter.[23] While it is beyond the scope of this paper to allow the firm full flexibility over maturity structures, our analysis in this section offers new insight on debt maturity. We first show if the firm is not constrained to borrow a fixed amount, then shareholders are indifferent to the maturity structure of any *future* debt issuance. While different maturity choices will lead to different future leverage levels, any increase in tax benefits is offset by an increase in default costs, and the firm's current share price is unaffected. But although shareholders are indifferent, a social planner that puts more weight on the dead weight costs of default would prefer that firms use long-term debt. On the other hand, a firm that needs to borrow a large amount quickly, or that wants to maximize its debt capacity, would choose short-term debt.

[23] Debt retirement in this fashion is similar to a sinking fund that continuously buys back debt at par; see Leland (1998), Leland and Toft (1996), He and Xiong (2012), and Diamond and He (2014). See, however, Brunnermeir and Oehmke (2013) and He and Milbradt (2016) for analysis of the firm's decision to lengthen or shorten its debt maturity.

Optimal Debt Maturity: Shareholder Indifference

Recall from **PROPOSITION 2** that we can compute the current value of equity as though the firm will not issue or repurchase debt in the future, and just repays its existing debt as it matures. This result immediately implies that for an initially unlevered firm ($F_0 = 0$), firm value does not depend on the choice of debt maturity structure ξ. This indifference result is deeper than it appears: Although the firm starts with no initial debt, recall that (41) says the firm will begin issuing debt immediately, and the debt maturity ξ does affect future debt contracts, leverage, and share prices. Nevertheless, because any gains from tax savings are offset by increased bankruptcy costs, these dynamic consequences have no effect on the initial share price, and as a result shareholders are indifferent across alternative debt maturity choices.

This irrelevance result can be generalized further. Consider the following thought experiment, in which equity – facing the current cash flows and debt structure $\left(Y_t, F_t, \xi\right)$ – has a one-time opportunity to choose an alternative maturity ξ' for the firm's *future* debt. That is, the firm's existing debts continue to retire at the old speed ξ, but the newly issued debts are with the new maturity and hence will retire at the new speed ξ'. We have the following proposition.

> **PROPOSITION 9.** In a no-commitment equilibrium with smooth debt issuance, the firm's current equity value is independent of the maturity ξ' of new debt.

PROOF: We can consider the implied future liabilities from the firm's existing debt as a modification of the cash flow process for the firm and then apply our general results from Section 2. For equilibria in which equity holders are taking smooth debt issuance polices, equity holders obtain zero profit by issuing future debt, and their value will be the same as if equity does not issue any future debt. As a result, the current equity value only depends on the maturity structure ξ of existing debt, but not on the maturity structure ξ' of future debt. ■

The logic of Proposition 9 and hence the indifference result can be further generalized to a setting in which the firm is free to choose any maturity structure for its newly issued debt any time. Again, the equity value only depends on the maturity structure of existing debt.

Although equity holders are indifferent between alternative maturity structures, different maturity choices will lead to very different patterns and levels of future leverage. For example,

Figure 4 shows the total enterprise value, debt amount, and leverage (debt value/TEV) for the firm given the same history of productivity shocks, but financed using either five-year debt (left panel) or one-year debt (right panel). In both cases the initial TEV and equity value is the same, but leverage evolves quite differently. With longer-term debt in the left panel, debt changes gradually, as the firm issues debt more slowly, and leverage is lower on average. With shorter-term debt, the firm issues debt more rapidly knowing it can decrease debt quickly by not rolling over maturing debt. Because it can adjust debt more quickly, the firm has higher leverage on average.[24]

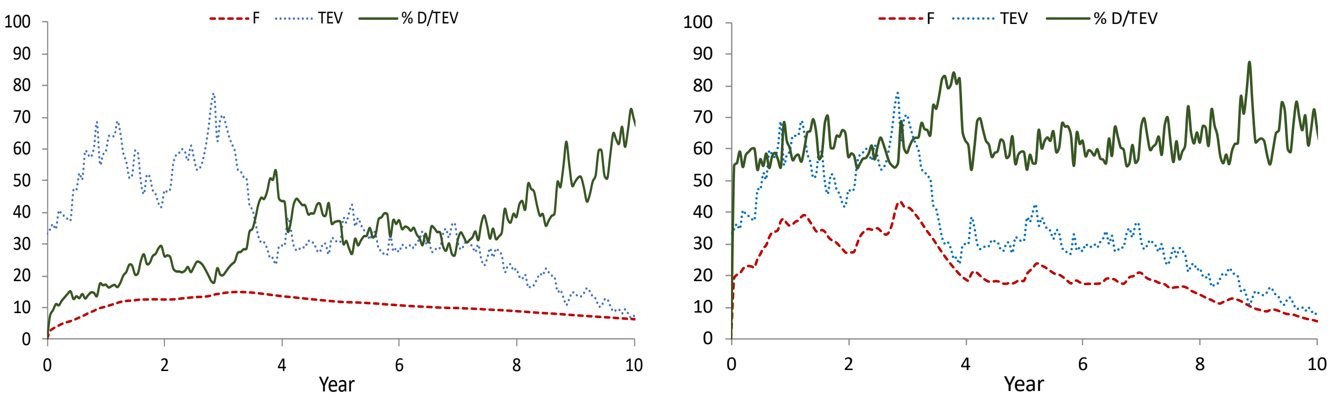

Figure 4: Debt and Leverage with Differing Maturities.
Left panel shows TEV, debt face value, and market leverage with 5-year average debt maturity. Right panel shows 1-year average debt maturity. In either case, initial firm value is unchanged, but leverage is higher overall and adjusts more quickly with shorter-term debt. Parameters are $\mu = 2\%$, $\sigma = 40\%$, $\overline{\pi} = 30\%$, $c(1-\overline{\pi}) = r = 5\%$, $\xi = 0.2$ (5-year debt) or 1 (1-year debt).

Figure 4 provides a potential explanation for the finding in Lemmon, Roberts, and Zender (2008), that much of the cross-sectional variation in firms' capital structure is persistent and largely unexplained by observable characteristics. From the perspective of our model, small perturbations or frictions that may lead firms to pick differing initial maturity structures will lead over time to dramatically different leverage outcomes.

Ultra-Short-term Debt and Commitment

A common intuition from the agency literature is that shareholder-creditor conflicts are ameliorated with short-term debt. As our analysis shows, without commitment this result is not correct: the use of short-term debt induces the firm to lever more aggressively, and the agency costs resulting from the leverage ratchet effect do not disappear.

[24] Of course, in our model we have assumed away transactions costs associated with issuing or rolling over debt. Such considerations would make long-term debt less costly, as in Dangl and Zechner (2016).

To highlight this contrast, consider ultra-short-term debt which matures instantaneously (i.e. at interval *dt*), much like demand deposits. If the firm can adjust its leverage freely in response to the cash-flows shocks, then given coupon rate $c > r - \mu$ the firm could set $F_t \approx Y_t / c$ and avoid default while capturing the entire debt tax shield. Choosing the following issuance policy

$$d\Gamma_t = (\mu + \xi) F_t dt + \sigma F_t dZ_t \,, \tag{42}$$

which makes $dF_t = \mu F_t dt + \sigma F_t dZ_t$, would potentially prevent the scaled cash flow y_t from fluctuating over time. Essentially, this captures the flexibility advantage offered by short-term debt.[25]

Although the flexibility benefit does apply in our model, the above argument implicitly assumes that equity holders can commit to the first-best leverage policy. In our model, the inability to commit to certain future leverage policy matters in a significant way – equity holders continue to raise debt until the likelihood of default impacts its price. This point is highlighted in (42), which requires the firm to repurchase debt following negative cash-flow shocks $dZ_t < 0$, while in our model shareholders never find debt repurchases optimal. In the limit, even with instantaneously maturing debt, there is always a risk of bankruptcy in our model, and the implied bankruptcy cost offsets the tax benefit.[26]

Optimal Maturity from a Social Perspective

As shown in **Figure 4**, a shorter debt maturity leads firms to take on more debt and hence capture a larger tax shield. But because initial firm value is unchanged, this larger tax shield is offset by a higher expected bankruptcy cost. Therefore, though shareholders are indifferent to debt maturity, from a social perspective there is a clear ex ante ranking. Because only bankruptcy results in a real deadweight cost, a planner should favor debt with the longest possible maturity, as long-term debt leads firms to take on less debt on average.

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[25] See Tserlukevich (2008) for further elaboration of this point. The flexibility offered by short-term debt is also studied in a recent paper by Geelen (2017).

[26] It might seem that with instantaneously maturing debt, there is no opportunity for shareholders to exploit existing creditors by issuing more debt prior to maturity. But note that new borrowing could be done essentially simultaneously with the initial borrowing (the firm issues debt in the morning, then more debt at lunch time, …, and so on so forth). For a model exploring the agency cost associated with sequential rounds of simultaneous borrowing, see Bizer and DeMarzo (1992).

We can calculate the expected bankruptcy cost $BC(y)$, as a fraction of unlevered firm value ϕY_0, and study how it varies with debt maturity ξ. Starting with initial debt F_0, debt evolves according to $dF_t = (g(y_t) - \xi) F_t dt$ and the firm default occurs at τ_b with $Y_{\tau_b} = y_b F_{\tau_b}$, implying

$$BC(y) = \frac{1}{\phi Y_0} \mathrm{E}\left[e^{-r\tau_b} \phi Y_{\tau_b} \middle| Y_0 = yF_0, F_0 \right] = \frac{y_b}{y} \cdot \underbrace{\mathrm{E}\left[\exp\left(-\int_0^{\tau_b} \left(r + \xi - g(y_t) \right) dt \right) \middle| y_0 = y \right]}_{H(y)}$$

We then compute the expectation term denoted by $H(y)$ numerically based on the following ODE

$$\left[r + \xi - g(y) \right] H(y) = \left(\mu + \xi - g(y) \right) yH'(y) + 0.5\sigma^2 y^2 H''(y)$$

with boundary conditions $H(y_b) = 1$ and $H(y) \to ky$ when $y \to \infty$ for some constant $k > 0$.[27]

For illustration, we focus on the case of an initially unlevered firm ($F_0 = 0$ which corresponds to $y = \infty$). In this case, we have $BC = ky_b$, where both endogenous constants k and y_b depend on debt maturity ξ. **Figure 9** plots BC against ξ for two different levels of cash-flow volatility. First, the planner who aims to minimize bankruptcy cost prefers longer-term debt (lower ξ). Second, interestingly, a lower volatility gives rise to a greater bankruptcy cost. The key to this counter-intuitive result is the endogenous debt issuance policy. Shareholders in a firm with lower

[27] Bankruptcy costs cannot exceed the first-best firm value which is linear in y. Strictly speaking, $H(y) = ky + o(y)$ for $y \to \infty$.

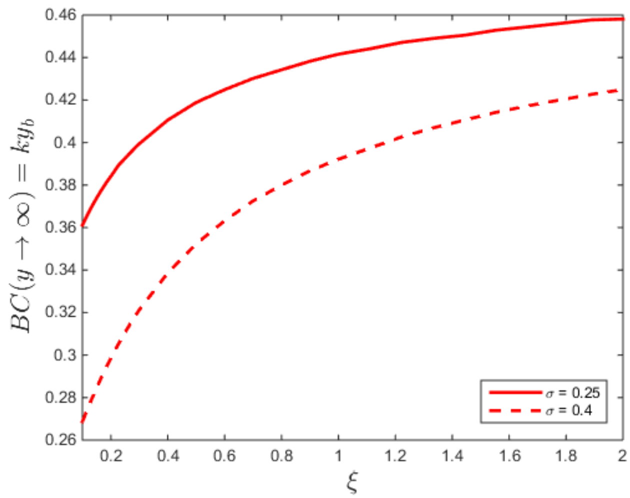

Figure 5. Expected bankruptcy costs *BC* (for an unlevered firm, as a fraction of unlevered firm value) as a function of debt retiring rate ξ. Bankruptcy cost is increasing in ξ, implying that the planner who minimizes bankruptcy cost prefers the longest-term debt. Two volatility levels are considered. Other parameters are $r = c(1-\bar{\pi}) = 5\%$, $\mu = 2\%$, $\bar{\pi} = 30\%$

volatility are more aggressive in levering up, so much so that we have a greater overall expected bankruptcy cost.

Another important advantage of studying an initially unlevered firm is that, in this case, expected bankruptcy costs move in tandem with expected tax shields.[28] This illustrates the sharp contrast between the objectives of the firm and that of the planner, as the shortest debt maturity maximizes tax shield but in the same time yields the largest bankruptcy cost. Note however, that this ex ante ranking can be reversed ex post, especially when the firm is close to default (y near y_b). Short-term debt (larger ξ) results in a lower bankruptcy costs $BC(y)$ once the firm is near distress because a firm with a larger ξ can delever more quickly (see **Figure 2**).

Maturity and Debt Capacity

Suppose the firm *must* raise some amount of funds initially through debt. We know from **PROPOSITION 1** that it is suboptimal to issue a discrete amount of debt in our model – shareholders would be better off issuing debt gradually – but suppose the firm must raise funds quickly and

[28] When $F_0 = 0$, the total enterprise value is $(1-\bar{\pi})Y_0/(r-\mu) = \phi Y_0$, independent of the debt maturity (recall the irrelevance result of debt maturity structure discussed before). Hence, the expected bankruptcy costs must equal the present value of future tax shields for all ξ.

equity capital is unavailable in the short run. In that case we can show that short-term debt maximizes not only the firm value, but also the debt capacity, i.e., the maximum amount of debt that the firm can raise.

Given initial cash-flow Y_0, the firm sets the initial debt face value F_0 to raise D_0 from debt holders. From (31) we know that total firm value is

$$\phi Y_0 \left[1 - \left(\frac{Y_0 / F_0}{y_b} \right)^{-\gamma - 1} \right] < \phi Y_0 . \tag{43}$$

Hence, both the total firm value and the debt value D_0 cannot exceed the upper bound ϕY_0.

PROPOSITION 10. Suppose the firm starts with initial cash flows Y_0.

i) The debt capacity, which is the highest debt value $\bar{D}_0$ that the firm can raise, is higher with a shorter debt maturity, and approaches ϕY_0 as $\xi \to \infty$.

ii) Given an fixed maturity ξ, firm value is decreasing in the amount of debt D_0 the firm must raise.

iii) For any feasible target debt value D_0, firm value will be higher with a shorter debt maturity (higher ξ).

PROOF: See the Appendix. ■

Figure 6 illustrates Proposition 10 by plotting the debt value (D_0) as a function of the debt face value (F_0) for different maturities (see top panel). Note that debt value declines if the face value is too high thanks to the "Laffer" curve effect: at some point an increase in the face value is more than offset by the increase in credit risk. The maximal debt capacity declines with maturity, as the credit risk of long-term debt is exacerbated by the firm's anticipated future debt issuance. With short-term debt, however, the firm is able to borrow close to the full value of the unlevered firm. Total firm value declines with the amount of upfront borrowing D_0, as shown in the bottom panel of Figure 6, and this effect is more severe when the debt has longer maturity.

Thus, a firm that is constrained to raise a discrete amount of debt quickly will find it optimal to use short-term debt. A key advantage of short-term debt is that by allowing it to retire, the firm can reduce leverage quickly should cash flows decline. This advantage of short-term debt is distinct relative to traditional Leland-type models in which the firm commits to maintain a fixed face value of debt (see Section 5 for a further analysis of such models). When the firm is committed to replacing retiring debt, it may suffer significant rollover losses (as the new debt sells at a discount to the face value the firm is repaying) which induce shareholders to default earlier. The shorter the maturity, the higher this rollover risk, making long-term debt optimal.[29]

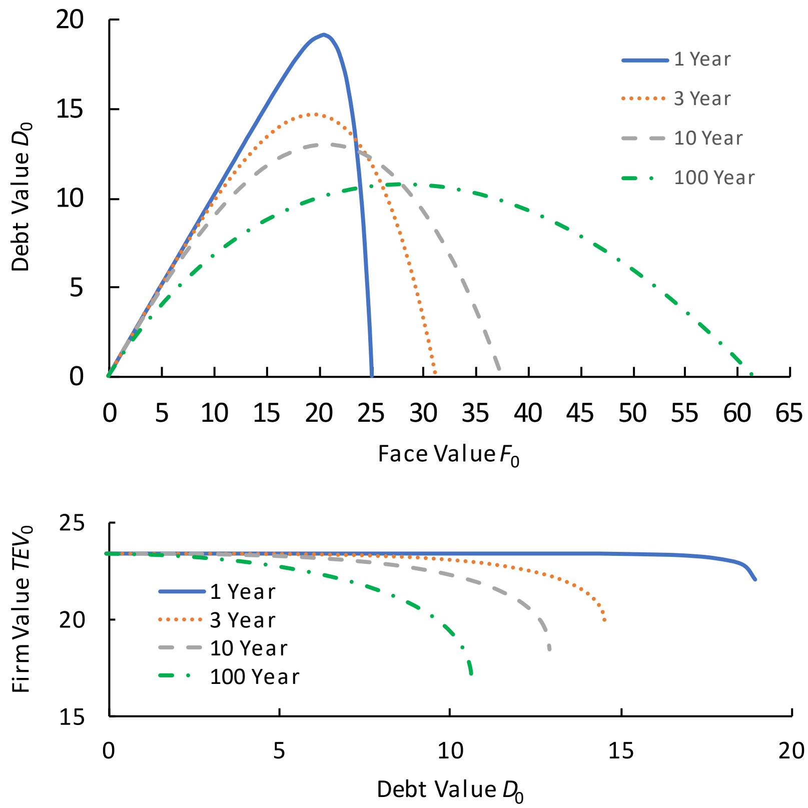

Figure 6. Debt Value and Firm Value for different levels of initial borrowing using alternative maturities. Debt value is hump-shaped in the face value of debt, and maximal debt capacity declines with maturity. Firm value declines with the level of initial borrowing as well as the maturity of the debt.
(Parameters are $r = c(1-\overline{\pi}) = 5\%$, $r = c(1-\overline{\pi}) = 5\%$, $\sigma = 40\%$, $\overline{\pi} = 30\%$, $\rho = 23.33$, $Y_0 = 1.0$.)

[29] See He and Xiong (2012) who show this result by having the debt face value fixed, and Diamond and He (2014) who strengthen this result by holding the debt market value fixed. Dangl and Zechner (2017) allow the firm to reduce leverage by not rolling over all debt; because debt issuance is costly in that model, intermediate term debt optimally trades off flexibility and transactions costs of refinancing.

5. Full Commitment Benchmarks

To facilitate discussion, we now solve two other cases that serve as benchmark with commitment. In each case, the firm's relevant state variables evolve exactly as in (21), except that the firm has the ability to commit to a certain future debt issuance policy.

No Future Debt Issues

We first consider the benchmark case in which the firm commits not to issue debt in the future, i.e. $g_t = 0$ always. We call this case the "No Future Debt" case, and indicate the corresponding solutions with the superscript "0" (representing the commitment to $g = 0$).

Recall that our methodology developed in Section 2 first called for solving the equity value function *as if* there will be no future debt issues, even though the firm will choose to add debt equilibrium, because the lack of commitment dissipates the benefits of debt tax shield. Therefore, from **PROPOSITION 2**, we have the equity value $v^0(y) = v(y)$ with the same default boundary y_b as calculated in **PROPOSITION 4.**

Indeed, the only change in this setting will be the debt pricing. Intuitively, new debt issues (despite the absence of dilution motive given zero recovery) harm existing creditors by accelerating default, and thus debt holders are willing to pay more for the same promise today if the firm can commit not to issue more debt in the future. With such a commitment, the firm's scaled cash flow evolves according to

$$\frac{dy_t}{y_t} = (\mu + \xi)\,dt + \sigma dZ_t\,, \tag{44}$$

and thus the HJB equation for debt price can be written as

$$\underbrace{rp^0(y)}_{\text{required return}} = \underbrace{c}_{\text{coupon}} + \underbrace{\xi\left(1 - p^0(y)\right)}_{\text{gain from principal repayment}} + \underbrace{(\mu + \xi)\,yp^{0\prime}(y) + \tfrac{1}{2}\sigma^2 y^2 p^{0\prime\prime}(y)}_{\text{evolution of } dy}, \tag{45}$$

with two boundary conditions. The first is no recovery value: $p^0\left(y_b^0\right) = 0$; and the second is risk-free pricing as the distance to default grows: $p^0(y) \to \dfrac{c+\xi}{r+\xi}$ as $y \to \infty$. Using standard methods (see e.g. the proof of **PROPOSITION 4**) the solution for the debt price is

$$p^0(y) = \frac{c+\xi}{r+\xi}\left[1-\left(\frac{y}{y_b}\right)^{-\gamma}\right], \tag{46}$$

where the constant γ is again given by (26). We summarize these results as follows:

PROPOSITION 11. If the firm can commit not to issue future debt, i.e. $g_t = 0$, then the equity value is unchanged, as is the default boundary, relative to the no commitment case. Default is delayed, however, and the debt price improves by the value of the debt tax shield

$$p^0(y) - p(y) = \underbrace{\frac{\bar{\pi}c}{r+\xi}\left[1-\left(\frac{y}{y_b}\right)^{-\gamma}\right]}_{\text{debt tax shield value}}. \tag{47}$$

PROOF: Equation (47) follows immediately from (28), (46) and $y_b^0 = y_b$. ■

From (47), we observe that when the firm can fully commit not to issue any debt in the future and hence is less likely to default, its debt will trade at a higher price than that of firms who cannot commit. Indeed, the premium is equal to the value of the tax shield, consistent with the observation that, in the no commitment case, the firm issues new debt at a rate so that expected bankruptcy costs offset the expected tax benefit. Thus, commitment to $g_t = 0$ does not benefit equity holders, but does improve the value of the debt due to the reduction in bankruptcy costs, which is just the expected tax benefit.

Fixed Face Value (Leland 1998)

Another relevant benchmark for our model without commitment is Leland (1998), who assumes that firm commits to keep a fixed total face value *F*. Specifically, in Leland (1998), the firm commits to replace the maturing debt (with intensity ξ) by the same amount of newly issued debt with the same coupon, principal, and maturity. We denote this case using the superscript "ξ", which requires $g_t = \xi$ always.

PROPOSITION 12. Suppose the firm commits to keep the face value of debt constant, i.e. $g_t = \xi$ always. Define

$$\gamma_p^\xi \equiv \frac{\mu - 0.5\sigma^2 + \sqrt{\left(\mu - 0.5\sigma^2\right)^2 + 2\sigma^2\left(r+\xi\right)}}{\sigma^2}, \text{ and } \gamma_v^\xi \equiv \frac{\mu - 0.5\sigma^2 + \sqrt{\left(\mu - 0.5\sigma^2\right)^2 + 2\sigma^2 r}}{\sigma^2}.$$

Then the value of equity and debt are given by

$$v^\xi\left(y\right) = \phi y + \frac{\overline{\pi} c}{r} - \left(\phi y_b^\xi + \frac{\overline{\pi} c}{r}\right)\left(\frac{y}{y_b^\xi}\right)^{-\gamma_v^\xi} - p^\xi\left(y\right), \text{ and } p^\xi\left(y\right) = \frac{c+\xi}{r+\xi}\left(1 - \left(\frac{y}{y_b^\xi}\right)^{-\gamma_p^\xi}\right).$$

with the default boundary, $\quad y_b^\xi = \dfrac{1}{(1+\gamma_v^\xi)\phi}\left[\dfrac{c+\xi}{r+\xi}\gamma_p^\xi - \dfrac{\overline{\pi} c}{r}\gamma_v^\xi\right].$

PROOF: See the Appendix. ■

Model Comparison: Valuation and Debt Issuance

Figure 7 plots the equity value and debt issuance policies for the three models: the base model without commitment ($g = g^*$, solid thick line), full commitment to no future debt ($g = 0$, dashed thin line), and Leland (1998) commitment to a fixed face value ($g = \xi$, dash-dotted thin line).

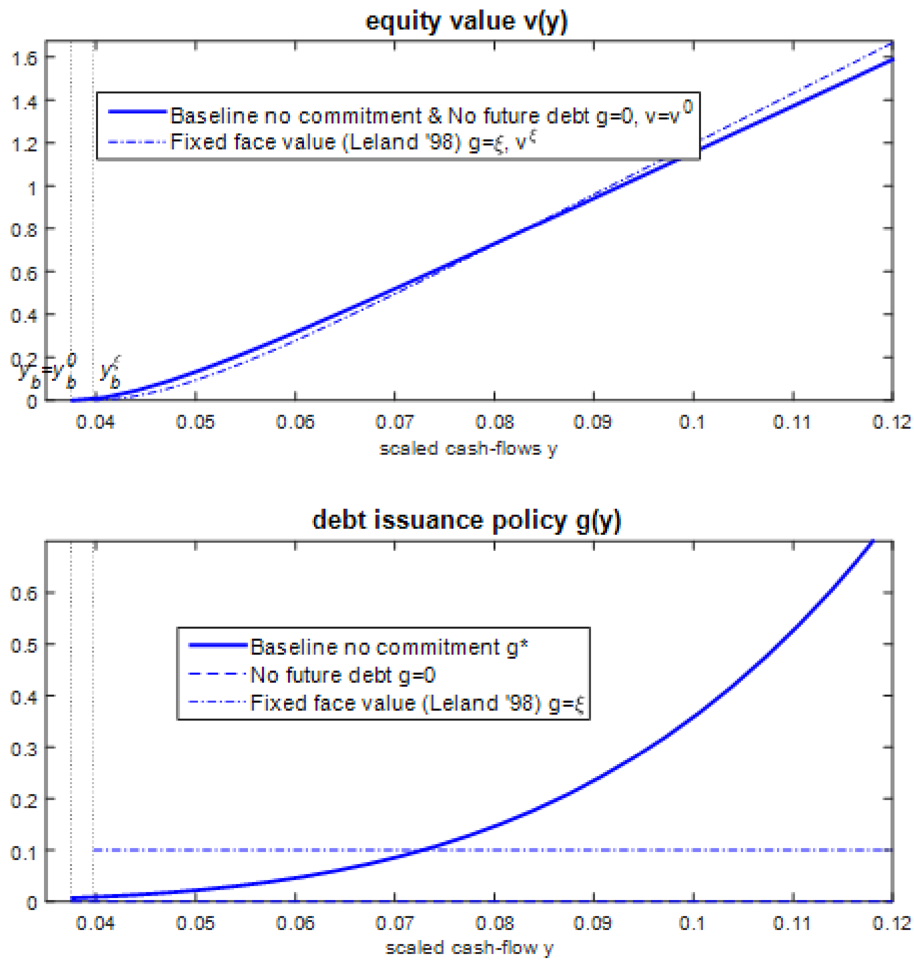

Figure 7: Equity Values for Alternative Debt Issuance Policies

Parameters are $r = 5\%$, $c = 8\%$, $\xi = 0.1$, $\mu = 2\%$, $\sigma = 25\%$, $\overline{\pi} = 35\%$.

As explained, the equity value in the no-commitment case coincides with the setting when there are no future debt issues. With a fixed face value policy, the equity value is lower when cash flows are low, as the firm is committed to continuing to issue debt even in the face of large rollover losses. This effect gives rise to a higher default boundary y_b^ξ than the default boundary in the other two cases $y_b = y_b^0$, as indicated in the top plot of Figure 7.[30] On the other hand, when cash flows

[30] However, $y_b^\xi < y_b$ could potentially occur, especially when the tax benefit $\overline{\pi}$ is high. Recall y_b is the default policy as if equity holder obtains no tax benefit, while for y_b^ξ, the firm with fixed debt face value indeed captures some tax benefit (hence, a high $\overline{\pi}$ pushes equity holders to default later).

are high, the equity value in the fixed face value case is higher than that in either case with $g = 0$ or $g = g^*$. Relative to the $g = 0$ case where tax benefits are lost as debt matures, the firm in the fixed face value case maintains its debt and hence enjoys greater tax benefits. On the other hand, the firm in the fixed face value case commits to a debt policy that is much less aggressive than the no-commitment case, hence incurring a much lower bankruptcy cost.

Figure 8 illustrates the debt price and valuation multiple for each policy. Not surprisingly, as shown in **PROPOSITION 11**, the debt price with "no future debt issuance" $g_t = 0$ dominates that without commitment, simply because future debt issuance pushes the firm closer to the default boundary. This also explains why in the bottom panel, the TEV multiple without commitment is always lower than that under commitment of $g_t = 0$ (recall equity values are the same under these two cases).

From (47), we see that the debt price premium due to "no future debt issuance" grows with the firm's distance to default. This implies that the debt of firms that cannot commit will exhibit large credit spreads even when the firm's current total leverage is very low--but, of course, the future leverage might be high. In fact, even for almost zero current leverage, the credit spreads for firms without commitment are non-zero. In contrast, the credit spreads for almost zero leverage firms are zero for the other two benchmark cases.

Relative to our base case, the fixed face value (Leland 1998) case generates a lower debt price for low y but higher debt price for high y. This is due to the endogenous issuance policy g^* plotted in Figure 7. There, we observe that the debt issuance policy without commitment is increasing in y, and slower (faster) than the fixed face value policy when y is low (high), and investors price the debt in anticipation of these future leverage polices.

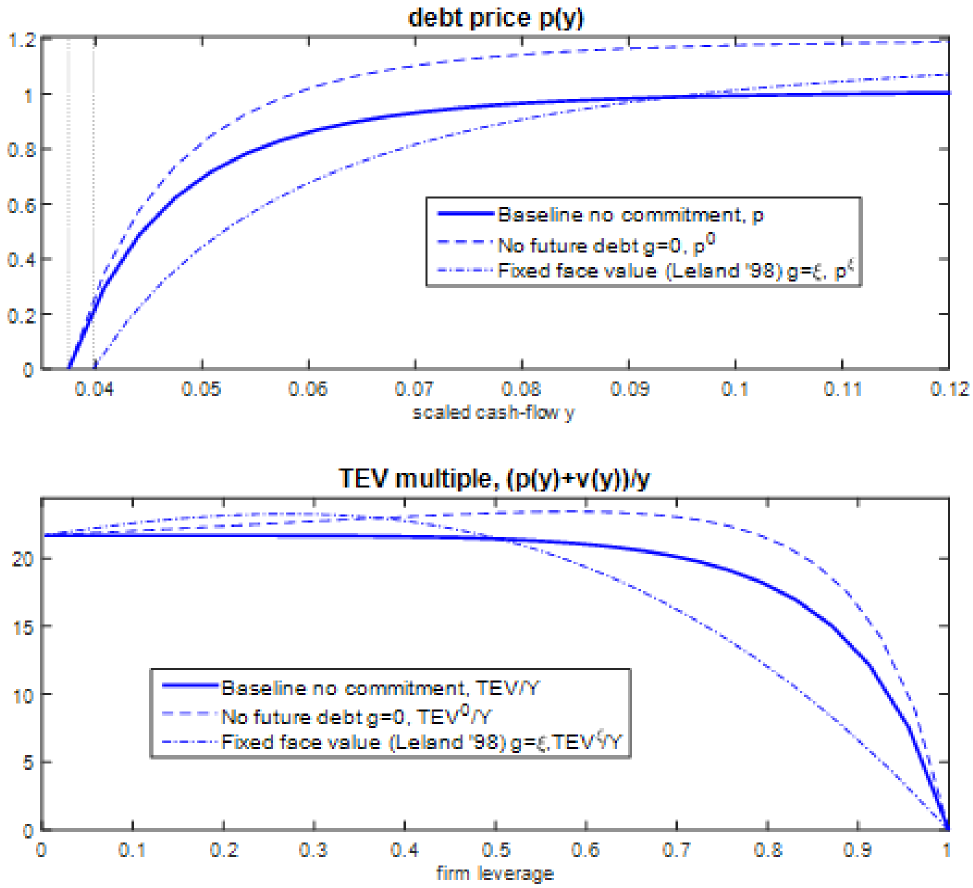

Figure 8: Debt Price and TEV Multiple for Alternative Debt Issuance Policies

Parameters are $r = 5\%$, $c = 8\%$, $\xi = 0.1$, $\mu = 2\%$, $\sigma = 25\%$, $\overline{\pi} = 35\%$.

The next proposition summarizes the comparison of debt values across three models, depending on the firm's profitability state y. Figure 8 corresponds to the case of $y_b^\xi > y_b$, so for sufficiently low y, the debt price in the case of fixed face value p^ξ drops below the other two cases.

Proposition 13. We always have $p(y) < p^0(y)$. For $y \to \infty$ we have

$$p(y) < p^\xi(y) < p^0(y).$$

For sufficiently low y so that $y \to \min(y_b, y_b^\xi)$, we have

$$p^\xi(y) < p(y) < p^0(y) \text{ if } y_b^\xi > y_b, \text{ and } p(y) < p^0(y) < p^\xi(y) \text{ if } y_b^\xi < y_b.$$

PROOF: See the Appendix. ■

Finally, as indicated in equation (31), in our no commitment case the firm's TEV multiple is strictly increasing in the scaled cash flow y. Consequently, holding the level of cash flows Y fixed, total firm value decreases with the debt face value F. In other words, in the no commitment equilibrium there is always a loss to total firm value from leverage – the tax benefits of debt more than offset the resulting bankruptcy costs due to future debt increases. This result is shown in the TEV-multiple-against-leverage plot in the bottom panel of Figure 8: there, the solid line (i.e., the no commitment case) achieves its maximum at zero leverage. In contrast, TEV multiples in both commitment cases have an interior maximum.[31] This interior maximum is often viewed as the "optimal" leverage in the traditional trade-off theory, though of course in a dynamic context it may be far from optimal ex-post once shocks are realized (see, e.g., Fischer, Heinkel, Zechner (1989) and Strebulaev (2007)).

Comparison of Leverage Dynamics

In two benchmark cases with commitment, the scaled cash-flows follow a geometric Brownian motion with exogenous drifts, i.e., $dy_t/y_t = (\mu+\xi)dt + \sigma dZ_t$ for the case of no future debt issuance, and $dy_t/y_t = \mu dt + \sigma dZ_t$ for the fixed face value case. In our model with no commitment, the equilibrium evolution of the firm's scaled cash-flows is:

$$dy_t = \left(\mu + \xi - g^*(y_t)\right) y_t dt + \sigma y_t dZ_t = \left[\mu + \xi - \frac{\pi c}{\rho\gamma}\left(\frac{y_t}{y_b}\right)^{\gamma}\right] y_t dt + \sigma y_t dZ_t . \tag{48}$$

The equilibrium debt issuance policy $g^*(y_t)$ in (30) is increasing in y_t, implying that y_t grows slower when y_t is higher. In fact, the firm's scaled cash-flows are mean-reverting towards the steady-state value (recall the definition of $\hat{y}_{\bar{g}}$ in (39)).[32]

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[31] This is evident in the bottom panel of Figure 8, as both dashed and dash-dotted lines have a positive slope at zero leverage, and drop to zero when leverage reaches 100%.

[32] Strictly speaking, to ensure y_t to mean revert over its equilibrium region $y_t \in [y_b, \infty)$, one has to show that $\hat{y}_{\mu+\xi} > y_b$ so that the drift of y_t is positive when $y_t = y_b$, which is indeed the case for our baseline parameters. However, $\hat{y}_{\mu+\xi} < y_b$ could occur for sufficiently large σ (so $\gamma \to 0$ in (49)).

$$\hat{y}_{\mu+\xi} \equiv y_b \left(\frac{\rho\gamma(\mu+\xi)}{\bar{\pi}c} \right)^{1/\gamma} \tag{49}$$

We are interested in the firm leverage dynamics implied by three different models. For given underlying cash-flow shocks $\{dZ_t\}$, the left panel of **Figure 9** plots the debt face value F_t, while the right panel shows the dynamics of scaled cash-flow y_t, which tracks one-to-one to the firm's interest-coverage-ratio (or book leverage). Because the underlying shocks are the same, the differences across these three different models are purely due to their different debt issuance policies. In this sample path, negative shocks in the early years cause the firm in our baseline no commitment case to issue less debt compared to the fixed face value case which commits to $g = \xi$, but of course since $g^* > 0$ the firm has more debt than it would in the no issuance case. Later, after a streak of positive shocks, the firm issues debt even faster than it matures and the debt level grows. As a result, y_t in the no commitment case (blue solid line) has a larger upward drift initially, but this reverses near the end of the sample path.

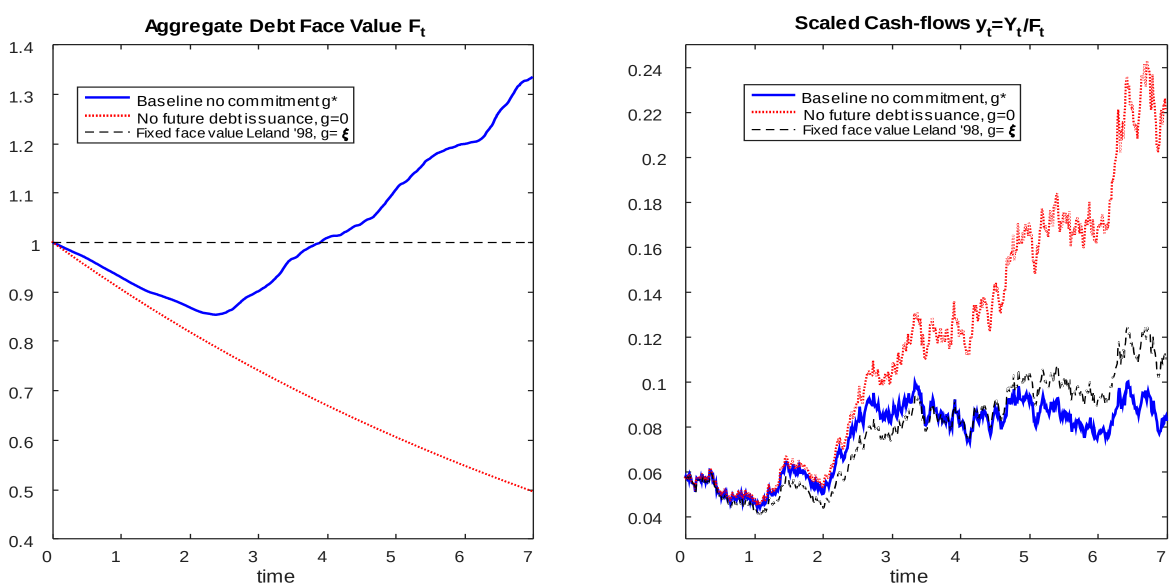

Figure 9: Aggregate debt face values (left panel) and scaled cash-flows dynamics (right panel) for three models.

With fixed cash-flow shocks $\{dZ\}$ and $r = 5\%$, $c = 8\%$, $\xi = 0.1$, $\mu = 2\%$, $\sigma = 25\%$, $\bar{\pi} = 35\%$.

6. Endogenous Investment and Debt Overhang

In this section, we extend our model by adding an endogenous investment decision also under the control of shareholders. Including investment allows us to explore the interaction of shareholder-creditor conflicts over investment and leverage choices. As expected from Myers (1977), and additional consequence of leverage is that debt overhang leads shareholders to underinvest. We show that when shareholders are unable to commit to future leverage decisions, the effect of debt overhang on investment is more severe when leverage is low and less severe when leverage is high compared to the case with a fixed debt level (as in Leland (1998)). On the other hand, debt overhang also impacts the leverage ratchet. Because underinvestment makes the debt price more sensitive to leverage, we find that for low leverage situations shareholders issue debt less aggressively than in the benchmark case in which investment is fixed. However, the leverage-ratchet effect becomes exacerbated near default, as shareholders reduce leverage less aggressively when they are able to cut investment.

6.1. General Analysis

Suppose equity holders may choose an endogenous investment policy i_t which enables the firm to increase the drift $\mu\left(Y_t, i_t\right)$ of the cash flow process at a cost $K\left(Y_t, i_t\right)$. Specifically, profitability evolves as[33]

$$dY_t = \mu\left(Y_t, i_t\right) dt + \sigma\left(Y_t\right) dZ_t, \tag{50}$$

and the firm generates cash at the rate[34]

$$\underbrace{Y_t}_{\text{operating cash flow}} - \underbrace{\pi(Y_t - cF_t)}_{\text{tax payment}} - \underbrace{\left(c+\xi\right) F_t}_{\text{debt interest \& principal}} - \underbrace{K(Y_t, i_t)}_{\text{capital investment}} + \underbrace{G_t p_t}_{\text{debt issuance/repurchase}} . \tag{51}$$

We assume both μ and K are smooth and increasing, and that μ is concave and K is convex in i.

[33] For illustration purposes, we consider a general cash-flow diffusion process without jumps; the analysis for a cash-flow process with jumps is similar.
[34] For simplicity we assume any tax consequences associated with investment are embedded in the cost function K.

As in the general analysis of Section 2, the HJB equation for equity is linear in the issuance rate *G*. As before we focus on the equilibrium where *G* takes interior solutions, which implies that $p = -V_F$. We can solve for the equity value function as if there is no debt issuance:

$$rV = \max_i \left[(1-\bar{\pi})Y - \left((1-\bar{\pi})c + \xi\right)F - \xi F V_F - K(Y,i) + \mu(Y,i)V_Y + \frac{1}{2}\sigma(Y)^2 V_{YY} \right]. \tag{52}$$

Optimal investment policy i^* is therefore characterized by the first-order condition

$$K_i\left(Y,i^*\right) = \mu_i\left(Y,i^*\right)V_Y . \tag{53}$$

The equilibrium issuance policy can be derived as before. Taking the derivative of (52) with respect to *F*, at the optimal investment policy i^*, and using $p = -V_F$, we have[35]

$$-rp = -((1-\bar{\pi})c + \xi) - \mu(Y,i^*)p_Y + \xi p + \xi F p_F - \frac{\sigma^2(Y)}{2} p_{YY} . \tag{54}$$

Comparing (54) with the valuation equation for the debt price in (11), we find the same debt issuance policy as in PROPOSITION 3**:**

$$G(Y,F) = \frac{\bar{\pi}c}{-p_F(Y,F)} = \frac{\bar{\pi}c}{V_{FF}(Y,F)} > 0 .$$

Again, the equilibrium requires the condition in PROPOSITION 1, i.e. $-p_F(Y,F) = V_{FF}(Y,F) > 0$ holds always.

6.2. Log-normal Cash Flows and Quadratic Adjustment Costs

Consider the setting with a log-normal cash-flow process studied in Section 3, with

$$\mu(Y,i) = (\mu + i)Y \text{ and } K(Y,i) = 0.5\kappa i^2 Y , \text{ where the constant } \kappa > \frac{2(1-\bar{\pi})}{(r-\mu)^2} .$$

Here, we can interpret investment as increasing the scale of the firm, with an adjustment cost that is proportional to the current scale and quadratic in the speed of adjustment. Without debt, our model is similar to Hayashi (1982), with the optimal investment policy equal to

[35] Here we apply the envelope theorem to ignore the dependence of *i** on *F*, which readily applies if investment takes interior solutions (as we will assume throughout). Even if the optimal investment policy takes a binding solution, the same logic applies as long as the constraint does not depend on *F*.

$$i_U = r - \mu - \sqrt{(r-\mu)^2 - 2(1-\bar{\pi})/\kappa}\,, \tag{55}$$

implying an unlevered firm value of $\kappa\, i_U\, Y$.[36]

Now we derive the solution to our model without commitment. Denote the optimal investment rate by i_t^* so that the evolution of scaled cash-flow $y_t = Y_t / F_t$ is

$$\frac{dy_t}{y_t} = \left(\mu + i_t^* + \xi - g_t\right)dt + \sigma dZ_t\,.$$

Equity holders default when y_t hits the endogenous default boundary y_b. The scaled equity value $v(y)$ without debt issuance satisfies

$$(r+\xi)v(y) = \max_i\,(1-\bar{\pi})(y-c) - \xi - \frac{\kappa i^2}{2}y + (\mu + i + \xi)yv'(y) + \frac{1}{2}\sigma^2 y^2 v''(y)\,. \tag{56}$$

Given the optimal investment $i^*(y) = v'(y)/\kappa$, the above equation becomes

$$(r+\xi)v(y) = (1-\bar{\pi})(y-c) - \xi + \frac{y\left[v'(y)\right]^2}{2\kappa} + (\mu+\xi)yv'(y) + \frac{1}{2}\sigma^2 y^2 v''(y)\,, \tag{57}$$

with two boundary conditions: $v(y) = \kappa i_U y - \rho$ for $y \to \infty$, and $v(y_b) = 0$. The default boundary y_b is determined by the smooth-pasting condition $v'(y_b) = 0$.

Although we no longer have closed-form solution in the model with investment, we can solve (57) numerically. But to assure we have a valid equilibrium, we must verify the key condition of PROPOSITION 1, that the equity value is convex (or equivalently, the debt price is decreasing) in *y*. The next proposition shows that indeed this condition holds.

PROPOSITION 14. In the log-normal cash-flow model with quadratic investment costs, equity value is strictly convex, i.e., $v''(y) > 0$, so that the debt price is decreasing in debt

[36] The unlevered firm chooses i to maximize the constant growth perpetuity value $\left[(1-\bar{\pi})Y - 0.5\kappa i^2 Y\right]/(r-\mu-i)$, which has solution i_U and value $\dfrac{(1-\bar{\pi})Y - 0.5\kappa i_U^2 Y}{r-\mu-i_U} = \dfrac{(1-\bar{\pi})Y + 0.5\kappa i_U^2 Y}{r-\mu} = \kappa i_U Y$.

face value. This guarantees the optimality of smooth issuance policy and hence the investment policy $i^*(y)=v'(y)/\kappa$ and issuance policy $g^*(y)=\dfrac{\bar{\pi}c}{yp'(y)}>0$, together with the debt price $p(y)=yv'(y)-v(y)$, constitute an equilibrium.

PROOF: See the Appendix. ■

6.3. Optimal Investment and Leverage

Having solved for shareholders' endogenous investment $i^*(y)$ and debt issuance $g^*(y)$ policies, we can compare them to several benchmarks with alternative investment or debt issuance constraints in order to study the interaction between these two types of agency costs. In particular, we can hold debt fixed as in the Leland (1998) model and consider the consequence for the firm's endogenous investment policies, or we can hold investment fixed (for example, at the optimal unlevered policy (55)) and compare the endogenous debt issuance policies. Appendix 8.2 gives the details of solving these benchmark cases. Figure 10 compares the investment policies (left panel) and debt issuance polices (right panel) for the different scenarios. The resulting investment and debt issuance polices from **PROPOSITION 14** are shown as solid blue lines, labeled "i^*, no commitment g^*."

Consider first the firm's investment as a function of its scaled cash flow. The black dashed line labeled "i_U, no commitment g^*" graphs the investment if it were fixed at the optimal level for the unlevered firm. As expected, when shareholders can choose investment, their policy i^* shows underinvestment relative to this benchmark, with the degree of underinvestment increasing as the firm's scaled cash flows decline. This underinvestment is the well-known debt overhang effect (Myers 1977; Hennessey, 2004); in the extreme, equity holders choose to cease investing altogether when the firm is close to default.

We also show the investment policy that would arise with fixed leverage as in the Leland (1998), depicted as the red dash-dotted line labeled "i^*, Leland '98 $g=\xi$." Here the firm maintains a fixed face value of debt, but shareholders choose investment endogenously. Compared with our model where there is no commitment to a debt level, we see even greater underinvestment near default in the Leland setting. The reason is that in our model, the distressed firm will choose to allow leverage to decline (as we saw in Figure 2), reducing the likelihood of default and increasing the expected return on investment. On the other hand, when leverage is moderate or low, the firm in the Leland fixed-debt setting investments more than when debt is unrestricted, and even more than the unlevered optimum. The reason is that the Leland firm, by committing to a debt level, captures a positive benefit from the debt tax shield which raises its expected return on investment. To summarize, the no commitment leverage police g^* leads to greater underinvestment when leverage is low, and less underinvestment when leverage is high.

In the right panel of Figure 10 we compare the rate of debt issuance polices when investment is endogenous versus when investment is fixed at the unlevered optimal level i_U. When leverage is low to moderate, debt issuance is about 7.5% slower when investment is endogenous. This moderation is due to the increased price impact of debt issuance as creditors anticipate the decline in investment due to debt overhang. The comparison reverses when leverage is high and the firm approaches financial distress. In that case, shareholders' option to cut investment lowers the risk of default and so makes the debt price less sensitive to new issuance, exacerbating the leverage-ratchet effect. Overall, these effects imply that debt overhang slows the rate of mean-reversion of firm leverage from either extreme.

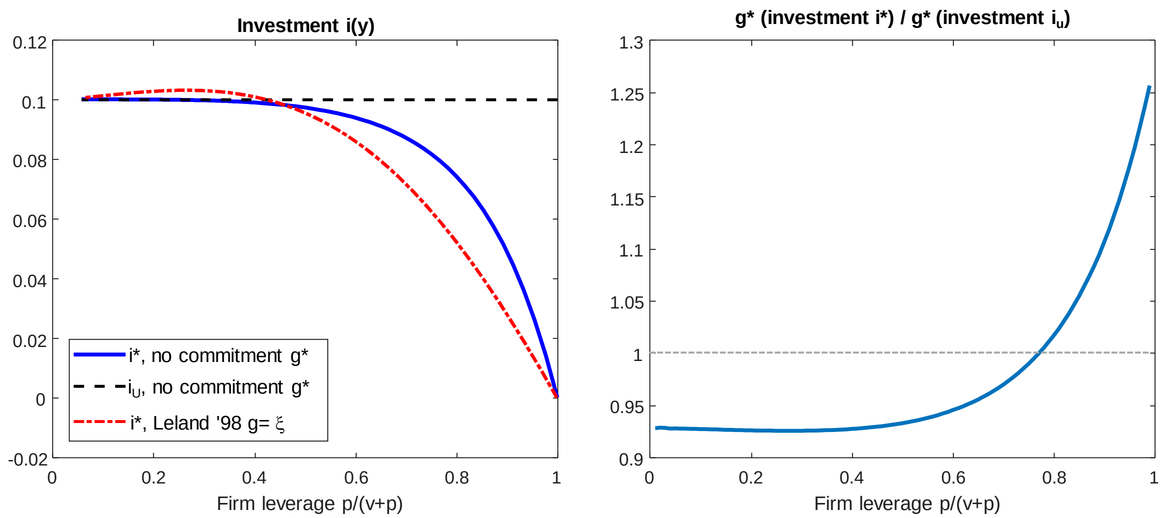

Figure 10 Endogenous investment and debt issuance policies.
Our model extension with endogenous investment is denoted by "i*, no commitment g*." The left panel compares investment with the first-best i_U and endogenous investment in the Leland (1998) setting with a fixed face value of debt. The right panel compares the speed of debt issuance when investment is endogenous versus when it is fixed at i_U. Parameters are $r = c(1-\overline{\pi}) = 5\%, \sigma = 40\%, \mu = -8\%, \xi = 0.2, \kappa = 87.5$ implying optimal growth $\mu + i_U = 2\%$ and steady-state leverage of approximately 40%.

7. Conclusions

When the firm cannot commit ex ante to future leverage choices, shareholders will adjust the level of debt to maximize the firm's current share price. As shown by Admati et al. (2017), capital structure decisions are then distorted and a leverage ratchet effect emerges: shareholders will choose to issue new debt gradually over time even if leverage is already excessive relative to the standard tradeoff theory optimum. This endogenous rate of debt issuance decreases as the firm approaches default, and is offset by the rate of asset growth and debt maturity, so that the firm's equilibrium leverage is ultimately mean-reverting.

We develop a general methodology to solve for equilibrium debt dynamics in this setting, including endogenous investment, and apply it to several standard models. When earnings evolve as geometric Brownian motion (including possible upward jumps), we explicitly solve for the firm's debt as a weighted average of past earnings, with the speed of adjustment decreasing with debt maturity and volatility.

Because creditors expect the firm to issue new debt in the future, credit spreads are wider in our model than in standard models with fixed debt, and remain wide even when firms are arbitrarily far from default. Lower debt prices dissipate the tax shield benefits of leverage, so that the equity value is identical to the case without no future debt issuance. This inability to capture tax benefits of leverage may provide a possible resolution for the zero-leverage puzzle (Strebulaev and Yang, 2013), as the potential tax benefit from leverage is offset by the high credit spread even for initial debt.

Finally, while shortening the maturity of future debt issues raises the average level of leverage as well as its speed of adjustment, it has no impact on the share price. As a result, even "instantaneous" debt does not resolve the agency problem, and equity holders have no incentive whatsoever to adjust the firm's debt maturity structure. This interesting observation also offers a potential explanation for the finding in Lemmon, Roberts, and Zender (2008), that much of the cross-sectional variation in firms' capital structure is persistent and largely unexplained by observable characteristics: Small perturbations or frictions that may lead firms to pick differing initial maturity structures will lead over time to dramatically different leverage outcomes.

Firms may try to reduce the agency costs resulting from the leverage ratchet effect by agreeing to covenants that restrict future debt issuance. Equity and debt issuance may also incur transactions costs or expose the firm to other market imperfections, which prompts the firm to actively manage its internal liquidity (cash) position (e.g., Bolton, Chen, and Wang, 2014). We leave for future work an exploration of the leverage dynamics that arise from the interaction of these additional forces with the leverage ratchet effects explored here.

8. Appendix.

8.1. Remaining Proofs

PROOF OF PROPOSITION 4. We can write the value function as

$$v(y) = \bar{v}(y) + E\left[e^{-(r+\xi)\tau_b}\right]\left(0 - \bar{v}(y_b)\right) = \underbrace{\bar{v}(y)}_{\substack{\text{equity value}\\\text{without}\\\text{default option}}} \underbrace{-\bar{v}(y_b)\left(\frac{y}{y_b}\right)^{-\gamma}}_{\text{option value of default}}, \tag{58}$$

where the expression for $E\left[e^{-(r+\xi)\tau_b}\right]$ and γ follows by solving the ODE

$$(r+\xi) f(y) = (\mu+\xi) y f'(y) + \tfrac{1}{2}\sigma^2 y^2 f''(y)$$

with boundary conditions $f(y_b) = 1$ and $f(\infty) = 0$. Finally, the optimal default boundary y_b is determined by the smooth-pasting condition, $v'\left(y_b\right) = 0$. ■

PROOF OF PROPOSITION 6. Note that the HJB equation (34) has the linear solution

$$\bar{v}\left(y\right) = \phi y - \rho. \tag{59}$$

The homogenous delayed differential equation

$$(r+\xi+\lambda) f(y) = (\hat{\mu}+\xi) y f'(y) + \tfrac{1}{2}\sigma^2 y^2 f''(y) + \lambda f(\theta y)$$

has solutions of the form $y^{-\hat{\gamma}}$ where $\hat{\gamma}$ solves the characteristic equation (35). In (35), because W is convex, $W\left(\infty\right) = W\left(-\infty\right) = \infty$, $W\left(0\right) < -r - \xi < 0$, and $W\left(-1\right) = \hat{\mu} - r < 0$, W has a unique positive real root (as well a unique negative real root $\hat{\eta} < -1$ that can be ruled out by the upper boundary condition). The remainder of the analysis follows exactly as in Section 3.2. ■

PROOF OF PROPOSITION 7. It is straightforward to check that (37) satisfies the HJB equation (24) or (34) with the boundary condition $v^L(y_b^L) = L(y_b^L)$ and smooth pasting condition $v^{L\prime}(y_b^L) = L'(y_b^L)$. The expressions for p^L and g^L follow from $p(y) = yv' - v$ and $g\left(y\right) = \dfrac{\bar{\pi} c}{y p'\left(y\right)}$ as in (28) and (30). ■

PROOF OF PROPOSITION 8: Using (30) the change in the face value of debt is (where we denote $\dot{F} \equiv \frac{dF_t}{dt}$)

$$\dot{F} = (g^*(Y/F) - \xi)F = \left(\frac{\bar{\pi}c}{\rho\gamma y_b^\gamma} Y^\gamma\right) F^{1-\gamma} - \xi F \,. \tag{60}$$

Let $H = F^\gamma$, then $\dot{H} = \gamma F^{\gamma-1}\dot{F}$ and so (60) implies

$$\dot{H} = \gamma F^{\gamma-1}\dot{F} = \gamma\left(\frac{\bar{\pi}c}{\rho\gamma y_b^\gamma} Y^\gamma\right) - \gamma\xi F^\gamma = \gamma\xi\left(\frac{Y}{\hat{y}_\xi}\right)^\gamma - \gamma\xi H \,.$$

Given H_0, this equation is a linear differential equation with general solution

$$H_t = e^{-\gamma\xi t}\left[H_0 + \int_0^t \gamma\xi\left(\frac{Y_s}{\hat{y}_\xi}\right)^\gamma e^{\gamma\xi s} ds\right].$$

(40) then follows from $F = H^{1/\gamma}$. ■

PROOF OF PROPOSITION 10: Given cash flows Y_0, and debt $F_0 \le Y_0 / y_b$, the market value D_0 of the firm's debt is

$$D_0(F_0) \equiv F_0\, p\left(y_0\right) = F_0\rho\left(1 - \left(\frac{y_0}{y_b}\right)^{-\gamma}\right) = \rho F_0\left(1 - \left(\frac{Y_0}{y_b}\right)^{-\gamma} F_0^\gamma\right). \tag{61}$$

The face value of debt which maximizes (61) is $F_0^* = \left(\frac{Y_0}{y_b}\right)(1+\gamma)^{-1/\gamma}$, with corresponding maximal debt capacity

$$\bar{D}_0 \equiv D_0(F_0^*) = \frac{\rho\gamma}{1+\gamma}\left(\frac{Y_0}{y_b}\right)(1+\gamma)^{-1/\gamma} = \phi Y_0 (1+\gamma)^{-1/\gamma} \,. \tag{62}$$

Result (i) then follows from the fact that (62) is increasing in γ, and γ is increasing in ξ (from (26)), and that $\xi \to \infty$ implies $\gamma \to \infty$, which implies $(1+\gamma)^{-1/\gamma} \to 1$.

From (31), total firm value is given by

$$TEV(F_0)=\phi Y_0\left[1-\left(\frac{y}{y_b}\right)^{-\gamma-1}\right]=\phi Y_0\left[1-\left(\frac{Y_0}{y_b}\right)^{-\gamma-1}F_0^{1+\gamma}\right], \tag{63}$$

which is decreasing in the F_0. Result (ii) follows because in the range $F_0\in[0,F_0^*]$, $D_0(F_0)$ is strictly increasing, and so to raise more funds D_0 the debt must have a higher face value. (Shareholders would never choose $F_0>F_0^*$, as they could raise more funds with a lower face value.)

For result (iii) we need to evaluate how total firm value changes with ξ for a fixed level of borrowing D_0. Combining (61) and (63) we have the following relation between D_0, γ, and TEV, and using $y_b=\frac{\gamma}{1+\gamma}\frac{\rho}{\phi}$, one solve for F_0 and show that

$$D_0=\phi Y_0\frac{1+\gamma}{\gamma}\left(\left(1-\frac{TEV}{\phi Y_0}\right)^{\frac{1}{1+\gamma}}-\left(1-\frac{TEV}{\phi Y_0}\right)\right)=\left(\phi Y_0-TEV\right)\frac{1+\gamma}{\gamma}\left(\left(\frac{\phi Y_0}{\phi Y_0-TEV}\right)^{\frac{\gamma}{1+\gamma}}-1\right) \tag{64}$$

Because γ is increasing in ξ, and TEV decreaseswith D_0, the result follows by showing that (64) is increasing in γ. To show this, letting $z\equiv\gamma/(1+\gamma)\in(0,1)$ and $\delta\equiv\frac{\phi Y_0}{\phi Y_0-TEV}>1$, the result is equivalent to showing that $\frac{1}{z}\left(\delta^z-1\right)$ is increasing in z, which follows from the fact that a compound returns grow faster than linearly. ■

PROOF OF PROPOSITION 12: The solution with constant face value is as follows. The scaled cash-flow y_t in this case follows

$$\frac{dy_t}{y_t}=\mu\,dt+\sigma dZ_t, \tag{65}$$

and equity holders in equilibrium will default at a threshold y_b^ξ to be derived shortly. Then, using the same logic as we did to compute p^0, we have the analogous solution to (46):

$$p^{\xi}(y)=\frac{c+\xi}{r+\xi}\left(1-\left(\frac{y}{y_b^{\xi}}\right)^{-\gamma_p^{\xi}}\right).$$

where the constant γ_p^{ξ} is defined by effectively lowering the drift by ξ (the rate of new debt issues) in (26):

$$\gamma_p^{\xi}=\frac{\mu-0.5\sigma^2+\sqrt{\left(\mu-0.5\sigma^2\right)^2+2\sigma^2\left(r+\xi\right)}}{\sigma^2}>0\ . \tag{66}$$

Next, the equity value $v^{\xi}(y)$ must solve

$$\underbrace{rv^{\xi}(y)}_{\text{required return}}=\underbrace{(1-\overline{\pi})(y-c)}_{\text{net after-tax earnings}}+\underbrace{\xi\left(p^{\xi}(y)-1\right)}_{\text{debt rollover gains/losses}}+\underbrace{\mu y v^{\xi\prime}(y)+\tfrac{1}{2}\sigma^2y^2v^{\xi\prime\prime}(y)}_{\text{evolution of state variable y}}, \tag{67}$$

with boundary conditions $v^{\xi}\left(y_b^{\xi}\right)=v^{\xi\prime}\left(y_b^{\xi}\right)=0$ and $v^{\xi}(y)\to\phi y+\frac{\overline{\pi}c}{r}-\frac{c+\xi}{r+\xi}$ as $y\to\infty$. Note that the second term in (67) captures the rollover gains/losses when equity holders refinance the maturing debt, as emphasized by He and Xiong (2012): per dollar of face value, the firm must repay principal at rate ξ, while equity holders commit to replace the maturing debt by issuing ξ new bonds at price p^{ξ}. Following Leland (1998), we can solve for the equity value function as

$$v^{\xi}(y)=\underbrace{\phi y+\frac{\overline{\pi}c}{r}}_{\substack{\text{Default-free}\\ \text{TEV}}}-\underbrace{\left(\phi y_b^{\xi}+\frac{\overline{\pi}c}{r}\right)\left(\frac{y}{y_b^{\xi}}\right)^{-\gamma_v^{\xi}}}_{\text{Cost of Default}}-\underbrace{\frac{c+\xi}{r+\xi}\left(1-\left(\frac{y}{y_b^{\xi}}\right)^{-\gamma_p^{\xi}}\right)}_{\text{Cost of Debt}} \tag{68}$$

where the constant γ_v^{ξ} is defined as

$$\gamma_v^{\xi}\equiv\frac{\mu-0.5\sigma^2+\sqrt{\left(\mu-0.5\sigma^2\right)^2+2\sigma^2r}}{\sigma^2}>0,$$

and the endogenous default boundary y_b^{ξ} satisfying the smooth-pasting condition $v^{\xi\prime}\left(y_b^{\xi}\right)=0$ is

$$y_b^{\xi}=\frac{r-\mu}{(1+\gamma_v^{\xi})(1-\overline{\pi})}\left[\frac{c+\xi}{r+\xi}\gamma_p^{\xi}-\frac{\overline{\pi}c}{r}\gamma_v^{\xi}\right].$$

■

PROOF OF PROPOSITION 13: $p(y) < p^0(y)$ is implied by (47). When $y \to \infty$,

$p^{\xi}(y) = \frac{c+\xi}{r+\xi}\left(1-\left(\frac{y}{y_b^{\xi}}\right)^{-\gamma_p^{\xi}}\right) \to \frac{c+\xi}{r+\xi}$ which exceeds

$p(y) = \frac{(1-\bar{\pi})c+\xi}{r+\xi}\left(1-\left(\frac{y}{y_b}\right)^{-\gamma}\right) \to \frac{(1-\bar{\pi})c+\xi}{r+\xi}$. To show that $p^{\xi}(y) < p^0(y)$, we need to show that $\left(\frac{y}{y_b}\right)^{-\gamma} < \left(\frac{y}{y_b^{\xi}}\right)^{-\gamma_p^{\xi}}$ holds when $y \to \infty$, which is equivalent to $\gamma_p^{\xi} < \gamma$; but the latter holds because γ increases with μ. The second part of result is obvious as the debt price drops to zero at the default boundary. ■

PROOF OF PROPOSITION 14: Define a constant $B \equiv \kappa i_U$ and $w(y) \equiv v(y) - By + \rho$; then $w(\cdot)$ is concave if and only if $v(\cdot)$ is concave. Using (57) and $v'(y) = w'(y) + B$, we have:

$$
\begin{aligned}
(r+\xi)w(y) &= (r+\xi)v(y) - (r+\xi)By + (1-\bar{\pi})c + \xi \\
&= (1-\bar{\pi})y + \frac{y\left(w'(y)+B\right)^2}{2\kappa} + (\mu+\xi)y\left(w'(y)+B\right) + \frac{\sigma^2 y^2}{2}w''(y) - (r+\xi)By \\
&= \underbrace{y\left[1-\bar{\pi}+\frac{B^2}{2\kappa}+(\mu+\xi)B-(r+\xi)B\right]}_{0,\text{ because } B=\kappa(r-\mu)-\sqrt{\kappa^2(r-\mu)^2-2\kappa(1-\bar{\pi})}} \\
&\quad + y\frac{2Bw'(y)+\left(w'(y)\right)^2}{2\kappa} + (\mu+\xi)yw'(y) + \frac{\sigma^2 y^2}{2}w''(y) \\
&= yw'(y)\left[\frac{B}{\kappa}+\mu+\xi\right] + y\frac{\left(w'(y)\right)^2}{2\kappa} + \frac{\sigma^2 y^2}{2}w''(y).
\end{aligned}
$$

Hence $w(y)$ satisfies the following ODE:

$$
(r+\xi)w(y) = yw'(y)\left[\frac{B}{\kappa}+\mu+\xi\right] + y\frac{\left(w'(y)\right)^2}{2\kappa} + \frac{\sigma^2 y^2}{2}w''(y) \tag{69}
$$

We need two steps to show that $w''(y) > 0$ for all $y > y_b$.

Step 1. $w(y)>0$ for all $y \geq y_b$. We know that at default $w'(y_b)=v'(y_b)-B=-B<0$, and $w(\infty)=0$. This implies that if $w(y)\leq 0$ ever occurs, then the global minimum must be nonpositive and interior. Pick that global minimum point y_1; we must have $w'(y_1)=0$ and $w''(y_1)>0$. Suppose that $w(y_1)<0$; evaluating (69) at y_1, we find that the LHS is strictly negative while the RHS is positive, contradiction. Suppose that $w(y_1)=0$; then there must exist some local maximum point $y_2>y_1$, so that $w(y_2)>0$, $w'(y_2)=0$ and $w''(y_2)<0$. But the same argument of evaluating (69) at y_2 leads to a contradiction.

Step 2. Because $w(y)$ approaches 0 from above when $y\to\infty$, we know that for y sufficiently large $w(y)$ is convex. Suppose counterfactually that $w(y)$ is not convex globally; we can take the largest inflection point y_2 with $w''(y_3)=0$. We must have $w'(y_3)<0$ and $w'''(y_3)>0$ (it is because for $y>y_3$ the function $w(y)$ is convex and decreasing to zero from above). At this point, differentiate (69) and ignore the term with $w''(y_3)=0$, and we have

$$\left(r-\mu-\frac{B}{\kappa}\right)w'(y_3)-\frac{\left(w'(y_3)\right)^2}{2\kappa}=\frac{\sigma^2 y_3^2}{2}w'''(y). \tag{70}$$

Recall $B=\kappa(r-\mu)-\sqrt{\kappa^2(r-\mu)^2-2\kappa}$ which implies

$$\left(r-\mu-\frac{B}{\kappa}\right)w'(y_3)=\sqrt{\kappa^2(r-\mu)^2-2\kappa}\cdot w'(y_3)<0$$

As a result, the LHS of (70) is negative while the RHS of (70) is positive, contradiction. This implies that $w(y)$ is convex globally.

Combining all the results above, we have shown that $v''(y)>0$ for $y>y_b$. ■

8.2. Appendix for Section 6.3

With slight abuse of notation, consider a constant investment policy i, which could take either the constant value of i_U in equation (55) or some other fixed level. The flow payoff to equity holders is $\left(1-\overline{\pi}\right)\left(y-c\right)-\frac{\kappa i^2}{2}y$, and the method in our base model allows us to derive the equity holders' value to be

$$v(y)=\frac{y(1-\overline{\pi}-\frac{\kappa i^2}{2})}{r-\mu-i}-\frac{c\left(1-\overline{\pi}\right)+\xi}{r+\xi}\left(1-\frac{1}{1+\gamma}\left(\frac{y}{y_b}\right)^{-\gamma}\right),$$

with endogenous default boundary

$$y_b=\frac{\gamma}{1+\gamma}\frac{r-\mu-i}{1-\overline{\pi}-0.5\kappa i^2}\left[\frac{c\left(1-\overline{\pi}\right)+\xi}{r+\xi}\right].$$

Then we can solve for the endogenous debt issuance policy g^* as in (30).

The solution to the Leland (1998) model with endogenous investment is characterized by a pair of ODE, one for the equity value $v^{\xi}\left(y\right)$ and the other for the debt price $p^{\xi}\left(y\right)$. For equity value, we have

$$rv^{\xi}\left(y\right)=\max_i\left(1-\overline{\pi}\right)\left(y-c\right)+\xi\left(p^{\xi}\left(y\right)-1\right)+\left(\mu+i\right)yv^{\xi\prime}\left(y\right)-\frac{\kappa i^2}{2}y+\tfrac{1}{2}\sigma^2y^2v^{\xi\prime\prime}\left(y\right)$$

With optimal investment $i^*\left(y\right)=\frac{v^{\xi}{}'\left(y\right)}{\kappa}$, the above ODE becomes

$$rv^{\xi}\left(y\right)=\left(1-\overline{\pi}\right)\left(y-c\right)+\xi\left(p^{\xi}\left(y\right)-1\right)+\mu yv^{\xi\prime}\left(y\right)+\frac{v^{\xi\prime}\left(y\right)^2}{2\kappa}y+\tfrac{1}{2}\sigma^2y^2v^{\xi\prime\prime}\left(y\right) \tag{71}$$

With boundary conditions $v^{\xi}\left(y_b\right)=0$, $v^{\xi}{}'\left(y_b\right)=0$, and $v^{\xi}{}'\left(y\right)=\kappa i_U$ for sufficiently large *y*.

For debt price $p^{\xi}\left(y\right)$, we have

$$\left(r+\xi\right)p^{\xi}\left(y\right)=c+\xi+\left(\mu+\frac{v^{\xi\prime}\left(y\right)}{\kappa}\right)yp^{\xi\prime}\left(y\right)+\tfrac{1}{2}\sigma^2y^2p^{\xi\prime\prime}\left(y\right) \tag{72}$$

with boundary conditions $p^{\xi}\left(y_b\right)=0, p^{\xi}{}'\left(y\right)=0$ for sufficiently large y. One can easily solve for $\left\{v^{\xi}\left(y\right),p^{\xi}\left(y\right)\right\}$ by solving the ODE system (71)-(72), with respective boundary conditions.

9. References

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