Internet Appendix to "Optimal Long-term Contracting with Learning"*

Zhiguo He† Bin Wei‡ Jianfeng Yu§ Feng Gao¶

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In this Internet Appendix (not for publication), Section A provides the detailed proof of Step 2 in proving Proposition 2 in the main text of the paper. We then study the case of non-stationary learning in which the underlying profitability θ is a parameter. We report the results of asymptotic analysis in Section C. Finally, in Section D we provide a detailed description of the algorithm used in solving the ordinary differential equation in the main text of our paper that characterizes the value function under the optimal contract, together with our Matlab programs that implement the algorithm.

A Step 2 in the Proof of Proposition 2

In this section, we provide detailed proof of step 2 as part of the proof of Proposition 2. Recall that the value function $V(p)$ satisfies the ODE (see Appendix A.7.1 in the main text of the paper):

$$rV(p) = \frac{1}{2}\frac{(1+p-\phi V_p(p))^2}{1+ar\sigma^2+a^2r^2\sigma^2\frac{(V_p(p))^2}{V_{pp}}} - p - \frac{1}{2}p^2 + V_p(p)(\phi+r)p, \quad (1)$$

$$s.t. V(\pm M_p) = H_1(\pm M_p). \quad (2)$$

Step 2: Under assumption 2, this ODE in equation (1) satisfies concavity and positivity of the denominator condition.

(a), Prove the concavity and positivity of the denominator at $p = 0$.

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†Corresponding author. University of Chicago, Booth School of Business, and NBER. 5807 South Woodlawn Ave., Chicago, IL 60637. Phone: (1)773-834-3769. Email: zhiguo.he@chicagobooth.edu.

‡Federal Reserve Bank of Atlanta, 1000 Peachtree St. N.E., Atlanta, GA 30309. Phone: (1)404-498-8913. Email: bin.wei@atl.frb.org.

§University of Minnesota, Carlson School of Management, CSOM 3-122, 321 19th Avenue South, Minneapolis, MN 55455, and PBC School of Finance, Tsinghua University, Beijing, China. Phone: (1)612-625-5498. Email: jianfeng@umn.edu.

¶Tsinghua University, School of Economics and Management, Beijing, China. Email: gaof@sem.tsinghua.edu.cn.

(b), Prove the concavity and positivity of the denominator for general $p > 0$.

(c), Prove the concavity and positivity of the denominator for $p < 0$.

(d), Prove that the lower boundary 0 is absorbing and the upper boundary $\bar{p}$ is entrance-no-exit.

We will use the results of optimal deterministic contract (i.e., the optimal contract within the space in which $\{\beta_t\}$ are deterministically time-varying) with value function $V^d(p)$ studied in Proposition 3 in the main text of the paper. Before we proceed to the proof of Step 2, we show the following lemma on the property of $\bar{p}$, which is useful in the subsequent proof.

Lemma A.1 *We have* $\overline{p} \leq \overline{p}^d$ *where* $\overline{p}^d \equiv B^d/A^d = \frac{2\phi}{(2\phi+r)ar\sigma^2+r+\sqrt{(2\phi+r)^2a^2r^2\sigma^4+2ar\sigma^2\left[(\phi+r)^2+\phi^2\right]+r^2}}$.

Proof. *From the key ODE equation (1), we know that at* $\overline{p}$*,*

$$V(\overline{p}) = H_2(\overline{p}) = \frac{1}{r}\left[\frac{1}{2}\frac{(1+\overline{p})^2}{1+ar\sigma^2} - \overline{p} - \frac{1}{2}\overline{p}^2\right].$$

Similarly, under the deterministic policy,

$$V^d\left(\overline{p}^d\right) = H_2\left(\overline{p}^d\right) = \frac{1}{r}\left[\frac{1}{2}\frac{\left(1+\overline{p}^d\right)^2}{1+ar\sigma^2} - \overline{p}^d - \frac{1}{2}\left(\overline{p}^d\right)^2\right].$$

It is straightforward that $H_2'(p) = \frac{1}{r}\left[\frac{1+p}{1+ar\sigma^2} - 1 - p\right] < 0$*, and hence to show* $\overline{p} \leq \overline{p}^d$*, it is enough to prove that* $V(p) > V^d(p)$*. Let's define function*

$$H(p) \equiv V(p) - V^d(p);$$

then the ODE equation (1) implies

$$rH(p) - p(\phi+r)H_p(p) = \frac{1}{2}\left[\frac{(1+p-\phi V_p(p))^2}{1+ar\sigma^2+a^2r^2\sigma^2\frac{V_p^2(p)}{V_{pp}(p)}} - \frac{\left(1+p-\phi V_p^d(p)\right)^2}{1+ar\sigma^2}\right]. \tag{3}$$

It follows from $\hat{x} = \frac{V_{pp}(0)-V_{pp}^d(0)}{1/\phi+A^d} > 0$ *that* $V_p(0+) > V_p^d(0+)$*, and hence initially* $V(p)$ *is above* $V^d(p)$*, i.e.,* $H(0+) > 0$ *and* $H(0) = 0$*.*

We show that $\overline{p} < \overline{p}^d$*. Suppose it is not true so that* $\overline{p} > \overline{p}^d$*. Since* V *is concave over* $[0, \overline{p}]$*,*

$$V_p\left(\overline{p}^d\right) > V_p(\overline{p}) = 0 = V_p^d\left(\overline{p}^d\right),$$

and hence,

$$H_p\left(\overline{p}^d\right) = V_p\left(\overline{p}^d\right) - V_p^d\left(\overline{p}^d\right) > 0.$$

In addition, if $\overline{p} > \overline{p}^d$*, then* $H\left(\overline{p}^d\right) = V\left(\overline{p}^d\right) - V^d\left(\overline{p}^d\right) < V(\bar{p}) - V^d\left(\overline{p}^d\right) = H_2(\bar{p}) - H_2\left(\overline{p}^d\right) < 0$*. Because* $H(0+) > 0$*, it must be that* V *crosses* V^d *at some* $\widehat{p}$ *before* $\overline{p}^d$*, such that*

$$H(\widehat{p}) = 0, \text{ and } H_p(\widehat{p}) < 0.$$

As a result, there exists another point $p_1 \in \left[\widehat{p},\ \bar{p}^d\right]$ *such that*

$$H(p_1) < 0,\ H_p(p_1) = 0 \text{ which implies that } V_p(p_1) = V_p^d(p_1).$$

However, it follows from equation (3) that at $p = p_1$,

$$\begin{aligned} rH(p_1) &= \frac{1}{2}\left[\frac{(1+p_1-\phi V_p(p_1))^2}{1+ar\sigma^2+a^2r^2\sigma^2\frac{V_p^2(p_1)}{V_{pp}(p_1)}} - \frac{\left(1+p_1-\phi V_p^d(p_1)\right)^2}{1+ar\sigma^2}\right] \\ &= \frac{(1+p_1-\phi V_p(p_1))^2}{2}\left[\frac{1}{1+ar\sigma^2+a^2r^2\sigma^2\frac{V_p^2(p_1)}{V_{pp}(p_1)}} - \frac{1}{1+ar\sigma^2}\right] > 0, \end{aligned}$$

which contradicts with $H(p_1) < 0$. ■

Step 2.a: Concavity & Positivity of the Denominator at $p = 0$ **:** We want to show that under assumption 2-3, the value function at $p = 0$ is concave, $V_{pp}(0) < 0$, and $1 + ar\sigma^2 + a^2r^2\sigma^2\frac{V_p(0)^2}{V_{pp}(0)} > 0$. Before we proceed to the main proof, we first show the following two lemmas, which are needed for our main proof. Define the following constants

$$\begin{aligned} A^d &\equiv \frac{(2\phi+r)\,ar\sigma^2 + r + \sqrt{(2\phi+r)^2 a^2r^2\sigma^4 + 2ar\sigma^2\left[(\phi+r)^2+\phi^2\right] + r^2}}{2\phi^2}, \\ A^o &\equiv r/\phi^2, \text{ and } B^d \equiv 1/\phi. \end{aligned}$$

Lemma A.2 *Denote*

$$\chi^o \equiv \frac{A^o}{1/\phi + A^o} \text{ and } \chi \equiv \frac{A^d - \frac{a^2r^2\sigma^2(1/\phi)^2}{1+ar\sigma^2}}{1/\phi + A^d}.$$

Under the condition

$$\frac{(r+\phi)^3}{\phi} < \frac{r^2}{2a}\left(1+ar\sigma^2\right), \tag{4}$$

for $a > 0$, *we have* $\chi^o < \chi$ *and*

$$A^d - A^o > \frac{ar\sigma^2}{\phi}\left(1+\frac{r}{\phi}\right) > \frac{2a^2\sigma^2\phi}{r(1+ar\sigma^2)}\left(1+\frac{r}{\phi}\right)^4. \tag{5}$$

Proof. Recall the definition of A^d and A^o, we have $A^o = \frac{r}{\phi^2}$, and

$$\begin{aligned} A^d &= \frac{1}{2\phi^2}\left((2\phi+r)\,ar\sigma^2 + r + \sqrt{(2\phi+r)^2 a^2r^2\sigma^4 + 2\sigma^2\left[(\phi+r)^2+\phi^2\right]ar + r^2}\right) \\ &= \frac{1}{\phi}\left(\frac{r}{2\phi}\left(1+ar\sigma^2\right) + ar\sigma^2 + \sqrt{\left[\frac{r}{2\phi}\left(1+ar\sigma^2\right)+ar\sigma^2\right]^2 + ar\sigma^2}\right). \end{aligned}$$

Because $\sqrt{\left(\frac{r}{2\phi}\left(1+ar\sigma^2\right)+ar\sigma^2\right)^2+ar\sigma^2} > \frac{r}{2\phi}\left(1+ar\sigma^2\right)+ar\sigma^2$, it follows that

$$A^d - A^o > \frac{1}{\phi}\left(\frac{r}{\phi}\left(1+ar\sigma^2\right)+2ar\sigma^2-\frac{r}{\phi}\right) > \frac{ar\sigma^2}{\phi}\left(1+\frac{r}{\phi}\right). \tag{6}$$

Furthermore, Assumption 2 (i.e., equation (4)) implies that

$$\begin{aligned} & \frac{\phi}{r}+a\phi\sigma^2 > 2a\left(\frac{\phi}{r}+1\right)^3 \Leftrightarrow \left(\frac{r}{\phi}\right)^3\left(\frac{\phi}{r}+a\sigma^2\phi\right) > 2a\left(1+\frac{r}{\phi}\right)^3 \\ \Leftrightarrow \quad & \left(\frac{r}{\phi}\right)^2\left(1+ar\sigma^2\right) > 2a\left(1+\frac{r}{\phi}\right)^3 \\ \Leftrightarrow \quad & \frac{ar\sigma^2}{\phi}\left(1+\frac{r}{\phi}\right) > \frac{2a^2\sigma^2\phi}{r\left(1+ar\sigma^2\right)}\left(1+\frac{r}{\phi}\right)^4. \end{aligned} \tag{7}$$

Thus, combining equation (6) and equation (7), we complete the proof for equation (5).

Lastly, we want to show $\chi^o < \chi$. First, the equality $2\left(1+\frac{r}{\phi}\right)^3 > \left(\frac{r}{\phi}\right)^3$ implies that

$$\frac{2a^2\sigma^2\phi}{r\left(1+ar\sigma^2\right)}\left(1+\frac{r}{\phi}\right)^4 > \left(1+\frac{r}{\phi}\right)\frac{a^2r^2\sigma^2\left(1/\phi\right)^2}{1+ar\sigma^2}.$$

Combining this with the equality in equation (5), we have

$$A^d > A^o + \left(1+\frac{r}{\phi}\right)\frac{a^2r^2\sigma^2\left(1/\phi\right)^2}{1+ar\sigma^2}.$$

Furthermore, notice that

$$\begin{aligned} \chi^o - 1 &< \chi - 1 \Leftrightarrow -\frac{1}{1+\phi A^o} < \frac{-1-\frac{a^2r^2\sigma^2(1/\phi)}{1+ar\sigma^2}}{1+\phi A^d} \\ A^d &> A^o + \frac{a^2r^2\sigma^2\left(1/\phi\right)^2}{1+ar\sigma^2}\frac{\left(1/\phi+A^o\right)}{1/\phi} \Leftrightarrow A^d > A^o + \frac{a^2r^2\sigma^2\left(1/\phi\right)^2}{1+ar\sigma^2}\left(1+\frac{r}{\phi}\right). \end{aligned}$$

Therefore, the inequality $\chi^o < \chi$ follows immediately. ■

Lemma A.3 *Under the condition in equation (4) and for* $a > 0$*, there exists a unique root* $x_0 \in (0, \chi^o)$ *to the following cubic polynomial:*

$$\begin{aligned} G(x) \equiv\ & \left(A^d - x\left(\frac{1}{\phi}+A^d\right)\right)(1-x)^2 + \frac{2x}{1+\phi A^d}\left(\left(A^d - x\left(\frac{1}{\phi}+A^d\right)\right)\left(1+ar\sigma^2\right)-a^2r^2\frac{\sigma^2}{\phi^2}\right) \\ & -\left(A^d - x\left(\frac{1}{\phi}+A^d\right)\right) + \frac{a^2r^2}{1+ar\sigma^2}\frac{\sigma^2}{\phi^2}. \end{aligned} \tag{8}$$

Proof. First, $G(0) = \frac{a^2r^2}{1+ar\sigma^2}\frac{\sigma^2}{\phi^2} > 0$. Second, notice that

$$\begin{aligned} G'(x) =\ & -\left(\frac{1}{\phi}+A^d\right)(1-x)^2 - 2\left(A^d - x\left(\frac{1}{\phi}+A^d\right)\right)(1-x) \\ & + \frac{2}{1+\phi A^d}\left(\left(A^d - x\left(\frac{1}{\phi}+A^d\right)\right)\left(1+ar\sigma^2\right)-a^2r^2\frac{\sigma^2}{\phi^2}\right) \\ & - \frac{2x}{1+\phi A^d}\left(\frac{1}{\phi}+A^d\right)\left(1+ar\sigma^2\right) + \left(\frac{1}{\phi}+A^d\right). \end{aligned}$$

Evaluating the above equation at $p=0$, we have

$$G'(0)=\frac{2A^d}{1+\phi A^d}\left(-\phi A^d+ar\sigma^2-\frac{a^2r^2\sigma^2}{A^d\phi^2}\right)-\frac{\frac{1}{\phi}+2A^d+\phi\left(A^d\right)^2}{1+\phi A^d}<0,$$

where the last inequality is due to the following fact

$$\phi A^d=\frac{r}{2\phi}\left(1+ar\sigma^2\right)+ar\sigma^2+\sqrt{\left[\frac{r}{2\phi}\left(1+ar\sigma^2\right)+ar\sigma^2\right]^2+ar\sigma^2}>ar\sigma^2.$$

Third, we proceed to show that $G''(x)>0$ for $x\leq\chi^o$. Note that the second derivative of $G(x)$ can be simplified to the following:

$$G''(x)=4\left(\frac{1}{\phi}+A^d\right)\left(1-x-\frac{1+ar\sigma^2}{1+\phi A^d}\right)+2\left(A^d-x\left(\frac{1}{\phi}+A^d\right)\right). \tag{9}$$

From Lemma A.2, we have $\chi^o<\chi$. Note that $\chi<\frac{A^d}{1/\phi+A^d}$, it follows that for $x\leq\chi^o<\chi$,

$$A^d-x\left(\frac{1}{\phi}+A^d\right)>0.$$

Furthermore, for $x\leq\chi^o$, we have

$$1-x-\frac{1+ar\sigma^2}{1+\phi A^d}\geq 1-\chi_0-\frac{1+ar\sigma^2}{1+\phi A^d}=\frac{\left(A^d-A^o\right)-\frac{ar\sigma^2}{\phi}\left(1+\frac{r}{\phi}\right)}{\left(1/\phi+A^o\right)\left(1+\phi A^d\right)}>0,$$

where the last inequality follows equation (5). Thus, both terms in equation (9) are positive, hence $G''(x)>0$.

Below we further show that $G(\chi^o)<0$ holds. After tedious algebra, one can show

$$\begin{aligned}G(\chi^o) &= (1/\phi)(2\chi^o)\left\{\frac{A^d-A^o}{1/\phi+A^o}\left[-1+\frac{\chi^o}{2}+\frac{1+ar\sigma^2}{1+\phi A^d}\right]+\frac{a^2r^2\sigma^2(1/\phi)}{(1+ar\sigma^2)}\left[\frac{1}{2\chi^o}-\frac{1+ar\sigma^2}{1+\phi A^d}\right]\right\}\\ &< (1/\phi)(2\chi^o)\left\{\frac{A^d-A^o}{1/\phi+A^o}\left[-1+\frac{\chi^o}{2}+\frac{1+ar\sigma^2}{1+\phi A^d}\right]+\frac{a^2r^2\sigma^2(1/\phi)}{(1+ar\sigma^2)}\frac{1}{2\chi^o}\right\}.\end{aligned}$$

It follows from straightforward algebra that

$$\begin{aligned}1-\frac{\chi^o}{2}-\frac{1+ar\sigma^2}{1+\phi A^d} &= \frac{\frac{r}{2\phi}+\sqrt{\left[\frac{r}{2\phi}+\frac{ar\sigma^2}{(1+ar\sigma^2)}\right]^2+\frac{ar\sigma^2}{(1+ar\sigma^2)^2}}}{\left(\frac{r}{2\phi}+1\right)+\sqrt{\left[\frac{r}{2\phi}+\frac{ar\sigma^2}{(1+ar\sigma^2)}\right]^2+\frac{ar\sigma^2}{(1+ar\sigma^2)^2}}}-\frac{\frac{r}{2\phi}}{1+\frac{r}{\phi}}\\ &> \frac{\frac{r}{2\phi}}{1+\frac{r}{2\phi}}-\frac{\frac{r}{2\phi}}{1+\frac{r}{\phi}}>\frac{\left(\frac{r}{2}\right)^2}{(\phi+r)^2}.\end{aligned}$$

Together with equation (5), we obtain

$$\begin{aligned}G(\chi^o) &< (1/\phi)(2\chi^o)\left\{-\frac{A^d-A^o}{1/\phi+A^o}\frac{\left(\frac{r}{2\phi}\right)^2}{\left(1+\frac{r}{\phi}\right)^2}+\frac{a^2r^2\sigma^2(1/\phi)}{(1+ar\sigma^2)}\frac{1+\frac{r}{\phi}}{2\frac{r}{\phi}}\right\}\\ &= -(1/\phi)\frac{\chi^o r^2\frac{1}{\phi}}{2\left(1+\frac{r}{\phi}\right)^3}\left\{\left(A^d-A^o\right)-\frac{2a^2\sigma^2\phi}{r(1+ar\sigma^2)}\left(1+\frac{r}{\phi}\right)^4\right\}<0.\end{aligned}$$

Now we have $G''(x) > 0$ for $x \le \chi^o$, $G'(0) < 0$, $G(0) > 0$ and $G(\chi^o) > 0$. Hence, there must exist one point $x \in (0, \chi^o)$ such that $G(x) = 0$. We still need to show the uniqueness. We consider two cases: $G'(\chi^o) \le 0$ and $G'(\chi^o) > 0$. For the first case, since $G''(x) > 0$ for $x \le \chi^o$, we have $G'(x) < 0$ for all $x \in (0, \chi^o)$, and hence $G(x)$ is monotonically decreasing in $(0, \chi^o)$. Thus, there is a unique x such that $G(x) = 0$. For the second case, since $G''(x) > 0$ for $x \le \chi^o$, there is a unique $x_1 \in (0, \chi^o)$ such that $G'(x_1) = 0$. For $x \in (0, x_1)$, $G(x)$ is strictly decreasing since $G'(x) < 0$ for $x \in (0, x_1)$. For $x \in (x_1, \chi^o)$, $G(x)$ is strictly increasing since $G'(x) > 0$ for $x \in (x_1, \chi^o)$. Thus, $G(x_1) < 0$ and there exists a unique $x \in (0, x_1) \subset (0, \chi^o)$ such that $G(x) = 0$. We complete our proof. ■

Given the above two lemmas, now we proceed to the main proof for step 2.a. Recall that the key ODE in equation (1) is

$$rV = \frac{1}{2}\frac{(1 + p - \phi V_p)^2}{1 + ar\sigma^2 + a^2r^2\sigma^2\frac{V_p^2}{V_{pp}}} - p - \frac{1}{2}p^2 + V_p(\phi + r)p.$$

The value function in the deterministic case V^d satisfies the following ODE (analyzed in Proposition 3 in the main text of the paper)

$$rV^d = \frac{1}{2}\frac{\left(1 + p - \phi V_p^d\right)^2}{1 + ar\sigma^2} - p - \frac{1}{2}p^2 + V_p^d(\phi + r)p.$$

Thus, taking the difference of the above two equations, we have

$$V - V^d = \frac{1}{2r}\left[\frac{(1 + p - \phi V_p)^2}{1 + ar\sigma^2 + a^2r^2\sigma^2\frac{V_p^2}{V_{pp}}} - \frac{\left(1 + p - \phi V_p^d\right)^2}{1 + ar\sigma^2}\right] + \left(\frac{\phi}{r} + 1\right)p\left[V_p - V_p^d\right].$$

For $p > 0$, dividing both sides by p and letting p go to zero, we have

$$\lim_{p\downarrow 0}\frac{V - V^d}{p} = \lim_{p\downarrow 0}\frac{1}{2rp}\left[\frac{(1 + p - \phi V_p)^2}{1 + ar\sigma^2 + a^2r^2\sigma^2\frac{V_p^2}{V_{pp}}} - \frac{\left(1 + p - \phi V_p^d\right)^2}{1 + ar\sigma^2}\right] + \left(\frac{\phi}{r} + 1\right)\lim_{p\downarrow 0}\left(V_p - V_p^d\right). \tag{10}$$

Because $V(0) = V^d(0) = 0$, we have

$$\lim_{p\downarrow 0}\frac{V - V^d}{p} = \lim_{p\downarrow 0}\frac{V - 0}{p} - \lim_{p\downarrow 0}\frac{V^d - 0}{p} = V_p(0) - V_p^d(0) = \lim_{p\downarrow 0}\left(V_p - V_p^d\right). \tag{11}$$

Thus, combining equation (10) and equation (11) yields

$$-\frac{\phi}{r}\lim_{p\downarrow 0}\left(V_p - V_p^d\right) = \lim_{p\downarrow 0}\frac{1}{2rp}\left[\frac{(1 + p - \phi V_p)^2}{1 + ar\sigma^2 + a^2r^2\sigma^2\frac{V_p^2}{V_{pp}}} - \frac{\left(1 + p - \phi V_p^d\right)^2}{1 + ar\sigma^2}\right].$$

Dividing both sides again by p and letting p go to zero, we have

$$-\frac{\phi}{r}\lim_{p\downarrow 0}\frac{V_p - V_p^d}{p} = \lim_{p\downarrow 0}\frac{1}{2rp^2}\left[\frac{\left(1+p-\phi V_p\right)^2}{1+ar\sigma^2+a^2r^2\sigma^2\frac{V_p^2}{V_{pp}}} - \frac{\left(1+p-\phi V_p^d\right)^2}{1+ar\sigma^2}\right]. \tag{12}$$

Noting that $V_p(0) = V_p^d(0) = \frac{1}{\phi}$, it follows from L' Hospital's rule that

$$\lim_{p\downarrow 0}\frac{V_p - V_p^d}{p} = \lim_{p\downarrow 0}\frac{V_p-(1/\phi)}{p} - \lim_{p\downarrow 0}\frac{V_p^d-(1/\phi)}{p} = V_{pp}(0) - V_{pp}^d(0), \tag{13}$$

and

$$\lim_{p\downarrow 0}\frac{\left(1+p-\phi V_p\right)^2}{p^2} = \lim_{p\downarrow 0}\frac{2\left(1+p-\phi V_p\right)\left(1-\phi V_{pp}\right)}{2p} = \left(1-\phi V_{pp}(0)\right)^2. \tag{14}$$

Further, plugging the expression for $V^d(p)$, we have

$$1+p-\phi V_p^d = 1+p-\phi\left(-A^d p + B^d\right) = \left(1+\phi A^d\right)p. \tag{15}$$

Substituting the above equations (13), (14), and (15) back into equation (12), we have

$$-\phi\left(V_{pp}(0) - V_{pp}^d(0)\right) = \frac{1}{2}\frac{\left(1-\phi V_{pp}(0)\right)^2}{1+ar\sigma^2+a^2r^2\sigma^2\frac{(1/\phi)^2}{V_{pp}(0)}} - \frac{\left(1+\phi A^d\right)^2}{2\left(1+ar\sigma^2\right)}.$$

Let's define constant $\hat{x}$ as

$$\hat{x} \equiv \frac{V_{pp}(0) - V_{pp}^d(0)}{\frac{1}{\phi}+A^d}.$$

Then, after tedious algebraic manipulation, one can show that $\hat{x}$ is a root of the 3-order polynomial $G(x)$, which is defined in equation (8) in Lemma A.3. According to Lemma A.3, there exists a unique positive root $x_0 \in (0, \chi^o)$. Thus, $\hat{x} = x_0$ is such a unique root. Further, Lemma A.2 implies $\chi^o < \chi < \frac{A^d}{1/\phi + A^d}$, and hence

$$0 < V_{pp}(0) - V_{pp}^d(0) = x_0\left(1/\phi + A^d\right) < \chi\left(1/\phi + A^d\right) < A^d.$$

Therefore,

$$V_{pp}(0) < V_{pp}^d(0) + A^d = -A^d + A^d = 0.$$

Furthermore, since $x_0 < \chi$, we have $x_0\left(1/\phi + A^d\right) - A^d < \chi\left(1/\phi + A^d\right) - A^d < 0$, and hence,

$$\begin{aligned}1+ar\sigma^2+a^2r^2\sigma^2\frac{V_p^2(0)}{V_{pp}(0)} &= 1+ar\sigma^2+a^2r^2\sigma^2\frac{(1/\phi)^2}{x_0\left(1/\phi+A^d\right)-A^d} \\ &> 1+ar\sigma^2+a^2r^2\sigma^2\frac{(1/\phi)^2}{\chi\left(1/\phi+A^d\right)-A^d} = 0.\end{aligned}$$

Thus, the denominator at $p = 0$ is always positive.

Step 2.b: Concavity & Positivity of the Denominator at $p > 0:$ We want to prove for any $p > 0$, the following must hold:

$$1 + ar\sigma^2 + a^2 r^2 \sigma^2 \frac{V_p^2(p)}{V_{pp}(p)} > 0 \text{ and } V_{pp}(p) < 0.$$

We prove the above inequalities by contradiction. Suppose that the above two inequalities fail to hold at some points. Denote the smallest point $\widehat{p} > 0$ at which at least one of the two inequalities does not hold. That is,

$$1 + ar\sigma^2 + a^2 r^2 \sigma^2 \frac{V_p^2(p)}{V_{pp}(p)} > 0 \text{ and } V_{pp}(p) < 0 \text{ for } p \in [0, \widehat{p})$$

while at $\widehat{p}$, one of the following three cases holds:

$$\text{(Case 1) } 1 + ar\sigma^2 + a^2 r^2 \sigma^2 \frac{V_p^2(\widehat{p})}{V_{pp}(\widehat{p})} = 0 \text{ and } V_{pp}(\widehat{p}) < 0$$

or

$$\text{(Case 2) } 1 + ar\sigma^2 + a^2 r^2 \sigma^2 \frac{V_p^2(\widehat{p})}{V_{pp}(\widehat{p})} = 0 \text{ and } V_{pp}(\widehat{p}) = 0$$

or

$$\text{(Case 3) } 1 + ar\sigma^2 + a^2 r^2 \sigma^2 \frac{V_p^2(\widehat{p})}{V_{pp}(\widehat{p})} > 0 \text{ and } V_{pp}(\widehat{p}) = 0.$$

We first show that Case 1 is impossible. Since $V_{pp}(p) < 0$ for $p \in [0, \widehat{p})$, we know that

$$T(p) \equiv 1 + p - \phi V_p \tag{16}$$

is strictly increasing in p and positive. To see this,

$$T'(p) = 1 - \phi V_{pp} > 0, \, T(0) = 0.$$

Therefore, $T(\widehat{p}) > 0$ and $T'(\widehat{p}) > 0$. This implies that

$$\frac{\left(1 + \widehat{p} - \phi V_p(\widehat{p})\right)^2}{1 + ar\sigma^2 + a^2 r^2 \sigma^2 \frac{V_p^2(\widehat{p})}{V_{pp}(\widehat{p})}} = \infty,$$

which contradicts the finiteness of V. Also we should use the fact that V_p is bounded due to $V \in \mathbb{C}^2$. By the same argument, Case 2 is also impossible.

We then consider Case 3. Note that Case 3 occurs only in the situation that

$$V_p(p) \to 0^+ \text{ and } V_{pp}(p) \to 0^- \text{ when } p \uparrow \widehat{p}.$$

Otherwise, a non-zero $V_p(\widehat{p})$ implies $\frac{V_p^2(\widehat{p})}{V_{pp}(\widehat{p})} = -\infty$, a contradiction to the fact that $1 + ar\sigma^2 + a^2r^2\sigma^2\frac{V_p^2(p)}{V_{pp}(p)} > 0$ for all $p < \widehat{p}$. Therefore, we can define a positive finite number $q \geq 0$ such that there exists some subsequence $\{p_n = \hat{p} - \epsilon_n\}$ approaching $\widehat{p}$ from left so that

$$\lim_{p_n \uparrow \widehat{p}} Q(p_n) \equiv \lim_{p_n \uparrow \widehat{p}} \frac{V_p^2(p_n)}{V_{pp}(p_n)} = -q \leq 0.$$

Moreover, we must have $\lim_{p \downarrow \widehat{p}} \frac{V_p^2(p)}{V_{pp}(p)} = -q \leq 0$ because otherwise $V(p)$ will exhibit a jump at $\widehat{p}$. Finally, we must have $1 + ar\sigma^2 - a^2r^2\sigma^2 q > 0$.

Taking the difference on the key ODE in equation (1) at $p = p_n$ and $p = \hat{p}$, for any sequence $\{p_n\} \to \hat{p}$ we have

$$\begin{aligned} & (1+\hat{p}-\phi V_p(\hat{p}))^2 - (1+p_n-\phi V_p(p_n))^2 \\ = \; & 2\left[rV(\hat{p}) + \hat{p} + \frac{1}{2}\hat{p}^2 - V_p(\hat{p})(\phi+r)\hat{p}\right]\left(1 + ar\sigma^2 + a^2r^2\sigma^2\frac{(V_p(\hat{p}))^2}{V_{pp}(\hat{p})}\right) \\ & -2\left[rV(p_n) + p_n + \frac{1}{2}p_n^2 - V_p(p_n)(\phi+r)p_n\right]\left(1 + ar\sigma^2 + a^2r^2\sigma^2\frac{(V_p(p_n))^2}{V_{pp}(p_n)}\right). \end{aligned}$$

Simplifying the above equation further, we obtain

$$\begin{aligned} (1+\hat{p}) \; = \; & (1+\hat{p})\left(1 + ar\sigma^2 + a^2r^2\sigma^2 Q(\hat{p})\right) + o_n(1) \\ & + \left[\frac{1}{2}\frac{(1+p_n-\phi V_p(p_n))^2}{1 + ar\sigma^2 + a^2r^2\sigma^2\frac{V_p^2(p_n)}{V_{pp}(p_n)}}\right]\left(a^2r^2\sigma^2\frac{Q(p) - Q(p_n)}{p - p_n}\right). \end{aligned}$$

Rearranging the above equation and noticing that $1 + ar\sigma^2 - a^2r^2\sigma^2 q > 0$, we have

$$arq - 1 = \frac{1}{2}\frac{1+\hat{p}}{1 + ar\sigma^2 - a^2r^2\sigma^2 q} ar\left(\lim_{n\to\infty}\frac{Q(\hat{p}) - Q(p_n)}{\hat{p} - p_n}\right). \tag{17}$$

Thus, $\lim_{n\to\infty}\frac{Q(\hat{p})-Q(p_n)}{\hat{p}-p_n}$ must exist and be finite.

First, suppose $q \neq 0$, then we know $\lim_{n\to\infty}\frac{V_{pp}(p_n)}{V_p^2(p_n)} = -\frac{1}{q}$, which is finite. Thus, it follows from $\frac{V_{pp}(\widehat{p}-\epsilon_n)}{V_p^2(\widehat{p}-\epsilon_n)} = -\frac{1}{q} + o_n(1)$ that

$$\begin{aligned} V_{pp}(p_n) \; & = \; -\frac{1}{q}V_p^2(p_n) + V_p^2(p_n)\, o_n(1) \\ & = \; -\frac{1}{q}2V_p(p_n)V_{pp}(p_n)\epsilon_n + 2V_p(p_n)V_{pp}(p_n)\epsilon_n o_n(1), \end{aligned}$$

where the second equality is due to the mean value theorem and $\hat{p} \geq \hat{p}_n \geq p_n$. Therefore, for any sequence $\{p_n\} \to \hat{p}$ we have

$$\begin{aligned} V_{ppp}(\widehat{p}) \; & \equiv \; \lim_{n\to\infty}\frac{V_{pp}(\widehat{p}-\epsilon_n)}{\epsilon_n} = \lim_{n\to\infty} -\frac{1}{q}2V_p(p_n)V_{pp}(p_n) + 2V_p(p_n)V_{pp}(p_n)\, o_n(1) \\ & = \; \lim_{n\to\infty} V_{pp}(p_n) = V_{pp}(\widehat{p}) = 0. \end{aligned}$$

Thus, $V_{ppp}(\widehat{p})$ exists, and hence using $Q(p) = \frac{V_p^2(p_n)}{V_{pp}(p_n)}$ equation (17) implies that

$$1 - arq = \frac{1}{2} \frac{1+\widehat{p}}{1 + ar\sigma^2 - a^2r^2\sigma^2 q} \left[ar \frac{V_p^2(\widehat{p})}{V_{pp}(\widehat{p})} \frac{V_{ppp}(\widehat{p})}{V_{pp}(\widehat{p})} \right]. \tag{18}$$

Below we show that there is a contradiction for Case 3 by the following three lemmas.

Lemma A.4 *It must be that either $q = 0$ or $q = \frac{1}{ar}$.*

Proof. *Assume that $q \neq 0$. Recall q is defined so that*

$$\lim_{p \uparrow \widehat{p}} \frac{V_p^2}{V_{pp}} = -q < 0,$$

which implies that

$$\lim_{p \uparrow \widehat{p}} \frac{V_p^2}{V_{pp}} = \lim_{p \uparrow \widehat{p}} \frac{2V_p V_{pp}}{V_{ppp}} = \lim_{p \uparrow \widehat{p}} 2V_p \cdot \frac{V_{pp}}{V_{ppp}} = -q,$$

Hence, we have

$$\lim_{p \uparrow \widehat{p}} \frac{V_{pp}}{V_{ppp}} = -\infty, \text{ and } \lim_{p \uparrow \widehat{p}} \frac{V_{ppp}}{V_{pp}} = 0^-$$

From equation (18) and $q \neq 0$, we have

$$\lim_{p \uparrow \widehat{p}} \frac{V_p^2}{V_{pp}} \frac{V_{ppp}}{V_{pp}} = 0,$$

and hence $q = \frac{1}{ar}$. As a result, $q = 0$ or $q = \frac{1}{ar}$. ■

Lemma A.5 *It is impossible to have $q = \frac{1}{ar}$.*

Proof. *Suppose $q = \frac{1}{ar}$. By plugging $V_p(\widehat{p}) = V_{pp}(\widehat{p}) = 0$ and $q = \frac{1}{ar}$ into the key ODE in equation (1), we obtain $V(\widehat{p}) = \frac{1}{2r} > V^{HM} = \frac{1}{2r}\frac{1}{1+ar\sigma^2}$, which, however, contractdicts the fact that the value function V^{HM} in the case of observable θ_t serves as an upper bound for the value function $V(p)$ (see the discussion before Proposition 2 in the main text of the paper).* ■

Lemma A.6 *It is also impossible to have $q = 0$.*

Proof. *Suppose that $q = 0$. From equation (17), we have*

$$-1 = \frac{1}{2} \frac{(1+\hat{p})}{1 + ar\sigma^2} \left(ar \lim_{n \to \infty} \frac{Q(\hat{p}) - Q(p_n)}{\hat{p} - p_n} \right),$$

which implies that $\lim_{n\to\infty} \frac{Q(\hat{p})-Q(p_n)}{\hat{p}-p_n} < 0$. Using this property, we can obtain a contradiction. To see this, $Q(\hat{p}) = \lim_{n\to\infty} \frac{V_p^2(p_n)}{V_{pp}(p_n)} = -q = 0$ at $\widehat{p}$. However, because $\widehat{p}$ is the first point so that $V_{pp}(\widehat{p}) = 0$, we know that $V_{pp}(\widehat{p}-) < 0$, which implies that

$$Q(\widehat{p}-) = \frac{V_p^2(\widehat{p}-)}{V_{pp}(\widehat{p}-)} < 0.$$

As a result, $\lim_{n\to\infty} \frac{Q(\hat{p})-Q(p_n)}{\hat{p}-p_n} > 0$, which is a contradiction. ■

Combining the above three lemmas, we obtain that Case 3 is impossible. Thus, we complete the proof for Step 2.b. Since $V(0) = 0$, $V_p(0) = \frac{1}{\phi} > 0$, and $V(M_p) < 0$, and $V(p) < 0$ for $p \geq 0$, it follows that there exists $\overline{p} > 0$ such that $V'(\overline{p}) = 0$. The analysis above also implies that $V_{pp}(\overline{p}) < 0$ strictly. Therefore, the function is strictly concave with a positive denominator for $p \in [0, \overline{p})$.

Step 2.c: Concavity & Positivity of the Denominator at $p < 0$ **:** Here we want to show that, for any $p < 0$,

$$1 + ar\sigma^2 + a^2r^2\sigma^2 \frac{V_p^2(p)}{V_{pp}(p)} > 0 \text{ and } V_{pp}(p) < 0.$$

Again, we prove these inequalities by contradiction, with a similar argument as in Step 2.b. Suppose that the above two inequalities fail to hold at some point. Denote the largest point $\widehat{p} < 0$ at which at least one of the two inequalities does not hold. That is,

$$1 + ar\sigma^2 + a^2r^2\sigma^2 \frac{V_p^2(p)}{V_{pp}(p)} > 0 \text{ and } V_{pp}(p) < 0 \text{ for } p \in (\widehat{p}, 0]$$

Because $V_{pp}(p) < 0$ for $p \in (\widehat{p}, 0]$ and $V_p(0) = \frac{1}{\phi} > 0$, we have

$$\begin{aligned} V_p(p) &> V_p(0) > 0 \text{ for } p \in (\widehat{p}, 0] \\ V_p(\widehat{p}) &\geq V_p(0) > 0. \end{aligned}$$

Similar to the argument in Step 2.b, one of the three cases below holds at $\widehat{p}$:

$$\text{(Case 1) } 1 + ar\sigma^2 + a^2r^2\sigma^2 \frac{V_p^2(\widehat{p})}{V_{pp}(\widehat{p})} = 0 \text{ and } V_{pp}(\widehat{p}) < 0$$

or

$$\text{(Case 2) } 1 + ar\sigma^2 + a^2r^2\sigma^2 \frac{V_p^2(\widehat{p})}{V_{pp}(\widehat{p})} = 0 \text{ and } V_{pp}(\widehat{p}) = 0$$

or

$$\text{(Case 3) } 1 + ar\sigma^2 + a^2r^2\sigma^2 \frac{V_p^2(\widehat{p})}{V_{pp}(\widehat{p})} > 0 \text{ and } V_{pp}(\widehat{p}) = 0.$$

We show that all three cases are impossible below.

Case 1. Suppose this is true. Then it must be true that

$$\frac{V_p^2(\widehat{p})}{V_{pp}(\widehat{p})} = -\frac{1 + ar\sigma^2}{a^2r^2\sigma^2}.$$

Moreover, for the value function to be bounded, it must be true that

$$1 + \widehat{p} - \phi V_p(\widehat{p}) = 0.$$

Since $V_p(\widehat{p}) \geq V_p(0)$ the above equation implies

$$\widehat{p} = \phi V_p(\widehat{p}) - 1 \geq 0,$$

which contradicts $\widehat{p} < 0$.

Case 2. This case is impossible, because $V_p(\widehat{p}) > 0$ and $V_{pp}(\widehat{p}) = 0$ imply $\lim_{p \downarrow \widehat{p}} \frac{V_p^2(\widehat{p})}{V_{pp}(\widehat{p})} = -\infty$, which contradicts with $1 + ar\sigma^2 + a^2r^2\sigma^2 \frac{V_p^2(\widehat{p})}{V_{pp}(\widehat{p})} = 0$.

Case 3. Note that Case 3 occurs only in the situation where

$$V_p(p) \to 0^+ \text{ and } V_{pp}(p) \to 0^- \text{ when } p \downarrow \widehat{p}.$$

Otherwise, a non-zero $V_p(\widehat{p})$ implies $\frac{V_p^2(\widehat{p})}{V_{pp}(\widehat{p})} = -\infty$, a contradiction to the fact that $1 + ar\sigma^2 + a^2r^2\sigma^2 \frac{V_p^2(p)}{V_{pp}(p)} > 0$ for all $\widehat{p} < p < 0$. On the other hand, as shown above $V_p(\widehat{p}) \geq V_p(0) > 0$, contradicting the fact that $V_p(p) \to 0^+$ when $p \downarrow \widehat{p}$.

Step 2.d: Absorbing lower boundary 0 & entrance-no-exit upper boundary $\overline{p}$:

Lemma A.7 *The lower bound $p = 0$ is absorbing. The upper boundary $\overline{p}$ is entry-no-exit, i.e., $\sigma_p = 0$ and $\mu_p < 0$. Therefore, $p_t \in [0, \overline{p}]$ always.*

Proof. We have shown that there exists an upper boundary $\overline{p}$ such that

$$V_p(\overline{p}) = 0.$$

This implies that $\beta(\overline{p}) = \frac{1+\overline{p}}{1+ar\sigma^2}$ and $\sigma^P(\overline{p}) = 0$, and

$$\begin{aligned} \left.\frac{dp}{dt}\right|_{p=\overline{p}} &= (\phi + r) p_t + \beta_t (ar\sigma\sigma^p - \phi) \\ &= (\phi + r) p_t - \beta_t \phi = \phi(\overline{p} - \beta(\overline{p})) + rp_t \\ &= \left(1 - r\overline{p}\frac{1}{\phi}\right) \frac{\phi}{(V_{pp}\phi - 1)} M. \end{aligned}$$

Since $V_{pp} < 0$, the above equation is negative if and only if $1 - r\overline{p}\frac{1}{\phi} > 0$. Hence, it follows from $A^d > r\left(\frac{1}{\phi}\right)^2$ (where $A^d > 0$ is the coefficient on the quadratic term for the deterministic value function), we have that

$$1 - r\overline{p}^d \frac{1}{\phi} = 1 - r\left(\frac{1}{\phi}\right)^2 \frac{1}{A^d} > 0.$$

Therefore, it follows from Lemma A.1 that $\overline{p} \leq \overline{p}^d$, and thus $1 - r\overline{p}\frac{1}{\phi} \geq 1 - r\overline{p}^d\frac{1}{\phi} > 0$.

Using a similar argument, we can prove that $p = 0$ is absorbing, since

$$\left.\frac{dp}{dt}\right|_{p=0} = \beta_t (ar\sigma\sigma^p - \phi) = 0.$$

■

This lemma is important because it implies that V is determined by the policy within the region $[0, \overline{p}]$. Because V is in C^2 and $V_{pp} < 0$ strictly, we know that the policy function

$$\beta(p) = \frac{1 + p - \phi V_p}{1 + ar\sigma^2 + a^2 r^2 \sigma^2 \frac{V_p^2}{V_{pp}}}$$

is bounded. Therefore, we can pick some arbitrary M to bound it.

B Appendix B: Non-stationary Learning

For tractability reasons we assume stationary learning in our model as in DeMarzo and Sannikov (2014). A non-stationary learning setting is also widely adopted when economic agents face parameter uncertainty (e.g., Prat and Jovanovic, 2014): the underlying profitability θ (as a parameter) never changes, and as time passes both parties eventually get to learn the true profitability. Does the front-loaded feature of the optimal effort profile depend on whether or not learning is stationary?

When learning is non-stationary, the posterior updating rules are

$$dm_t = \Sigma_t^\theta \frac{dY_t - (\mu_t + m_t)\,dt}{\sigma^2} \equiv \frac{\Sigma_t^\theta}{\sigma} dB_t^\mu, \text{ and } \Sigma_t^\theta = \frac{\sigma^2 \Sigma_0^\theta}{\sigma^2 + \Sigma_0^\theta t}, \tag{19}$$

The most salient difference lies in the conditional variance Σ_t^θ. In the stationary case that we are studying, the posterior variance is constant over time, while in the non-stationary case (e.g., Prat and Jovanovic, 2014), the posterior variance decreases over time and vanishes asymptotically.

If learning is non-stationary, on the equilibrium path eventually everyone learns θ perfectly, which implies that over time there are less and less noise in output dY_t. This is important, as optimal contracting is essentially an inference problem (Holmstrom, 1979). When time goes by, the signal-to-noise ratio of dY_t goes up, which implies that *information quality* goes up in designing the optimal contract. In other words, this information quality effect potentially pushes the optimal effort policy to be back-loaded (i.e., getting closer to the first-best level) to later periods with less uncertainty. This is an opposite force against the information rent effect, which favors front-loaded effort policies.[1]

Which force is stronger, the information rent effect or the information quality effect? The following proposition shows that, under deterministic policies, even with non-stationary learning the optimal effort policy still decreases with time, suggesting that the information rent effect dominates the information quality effect. We leave future research to investigate the general optimal contract with non-stationary learning.

Proposition B.1 *Restrict attention to the deterministic policies. With non-stationary learning so that the posterior variance follows equation (19), the optimal effort policy is decreasing over time.*

Proof. *We first formulate the problem. Suppose that the unknown parameter θ is a constant. Define $\phi_t \equiv \Sigma_t^\theta/\sigma^2$, hence*

$$dm_t = \Sigma_t^\theta \frac{dY_t - (\mu_t + m_t)\,dt}{\sigma^2} = \phi_t \sigma \frac{dY_t - (\mu_t + m_t)\,dt}{\sigma} \equiv \phi_t \sigma dB_t^\mu, \text{ and } \phi_t = \frac{\phi_0}{1 + \phi_0 t}, \tag{20}$$

[1] Although both parties learn profitability θ perfectly asymptotically so that we end up with the Holmstrom and Milgrom (1987) setting, the optimal contract might not implement the level implied by Holmstrom and Milgrom (1987) in the distant future, simply because it could leave too much information rents to the agent in earlier periods.

where we still use ϕ_t without risk of confusion. According to Prat and Jovanovic (2014), the agent's incentive-compatibility constraint is

$$\mu_t = \beta_t - \mathbb{E}_t \left[\int_t^T \phi_s \beta_s e^{-r(s-t)} e^{\left(-\int_t^s ar\beta_u \sigma dB_u - \frac{1}{2}\int_t^s a^2 r^2 \beta_u^2 \sigma^2 du\right)} ds \right] = \beta_t - \int_t^T \phi_s \beta_s e^{-r(s-t)} ds,$$

where in the second equation we invoke the restriction that $\{\beta\}$ are deterministic. Thus, the principal's problem can be written as (where T can take a value of infinity)

$$\max_{\{\beta_t\}} \int_0^T e^{-rt} \left(\mu_t - \frac{1}{2}\mu_t^2 - \frac{1}{2} ar\sigma^2 \beta_t^2 \right) dt$$

$$s.t. \qquad \mu_t = \beta_t - \int_t^T \phi_s \beta_s e^{-r(s-t)} ds.$$

Define $p_t \equiv \int_t^T \phi_s \beta_s e^{-r(s-t)} ds = e^{rt} \int_t^T \phi_s \beta_s e^{-rs} ds$ so that

$$\mu_t = \beta_t - p_t,\ p_t' = -\phi_t \beta_t + rp_t \Rightarrow \beta_t = -\frac{1}{\phi_t}\left(p_t' - rp_t\right).$$

Under this transformation, the objective becomes $\max_{\{p_t\}} \int_0^T L\left(t, p_t, p_t'\right) dt$, so that

$$L\left(t, p_t, p_t'\right) \equiv e^{-rt} \left(-\frac{1}{\phi_t}\left(p_t' - rp_t\right) - p_t - \frac{1}{2}\left(\frac{1}{\phi_t}\left(p_t' - rp_t\right) + p_t\right)^2 - \frac{ar\sigma^2}{2}\left(\frac{1}{\phi_t}\left(p_t' - rp_t\right)\right)^2 \right).$$

with the constraint that $p_T = 0$.

We now have a standard problem of calculus of variation, and the Euler equation for this problem is:

$$L_p\left(s, p_s, p_s'\right) = \frac{dL_{p'}\left(s, p_s, p_s'\right)}{ds}. \tag{21}$$

Because

$$\begin{aligned} L_p\left(s, p_s, p_s'\right) &= e^{-rs} \left(\frac{r}{\phi_s} - 1 + \left(\frac{1}{\phi_s}\left(p_s' - rp\right) + p_s \right) \left(\frac{r}{\phi_s} - 1 \right) + ar\sigma^2 \left(\frac{1}{\phi_s}\left(p_t' - rp\right) \right) \frac{r}{\phi_s} \right), \\ L_{p'}\left(s, p_s, p_s'\right) &= e^{-rs} \left[-\frac{1}{\phi_s} - \frac{1}{\phi_s}\left(\frac{1}{\phi_s}\left(p_s' - rp\right) + p_s \right) - ar\sigma^2 \left(\frac{1}{\phi_s}\right)^2 \left(p_s' - rp_s\right) \right], \end{aligned}$$

conducting algebraic simplifications on the Euler equation (21) yields

$$0 = -\frac{1}{\phi_s}\left(2p_s' - rp + \frac{1}{\phi_s}\left(p_s'' - rp_s'\right) \right) - 2ar\sigma^2 \left(\frac{1}{\phi_s}\right)\left(p_s' - rp_s\right) - ar\sigma^2 \left(\frac{1}{\phi_s}\right)^2 \left(p_s'' - rp_s'\right).$$

Therefore, from the above equality, we obtain that the optimal policy must satisfy:

$$\frac{1}{\phi_s} p_s'' = -2p_s' + \frac{1}{\phi_s} rp_s' + rp\left(1 + \frac{ar\sigma^2}{1 + ar\sigma^2}\right). \tag{22}$$

Combining with $p_T = 0$, one can solve the optimal path using initial condition p_0. Then, maximizing over p_0, one can find the solution to the original problem.

Before we proceed to prove the main result in the next subsection, we further show that the first-order optimality of the Euler equation is sufficient for global optimality. Notice that $L_{pp} < 0$ and $L_{p'p'} < 0$. Furthermore, for any x and y, we aim to show that

$$L_{pp}y^2 + 2L_{pp'}xy + L_{p'p'}x^2 \leq 0, \tag{23}$$

where the equality holds only for $x = y = 0$. Notice that

$$e^{rs}\left[L_{pp}y^2 + 2L_{pp'}xy + L_{p'p'}x^2\right] = -\left(\left(\frac{r}{\phi_s} - 1\right)^2 + ar\sigma^2\left(\frac{r}{\phi_s}\right)^2\right)y^2$$
$$+2xy\left(\left(\frac{1}{\phi_s}\left(\frac{r}{\phi_s} - 1\right) + ar\sigma^2\left(\frac{r}{\phi_s}\right)^2\right)\right) - x^2\left(1 + ar\sigma^2\right)\left(\frac{1}{\phi_s}\right)^2.$$

Thus, it is sufficient to show the following:

$$\left[\left(\frac{r}{\phi_s} - 1\right) + ar\sigma^2\frac{r}{\phi_s}\right]^2 < \left[\left(\frac{r}{\phi_s} - 1\right)^2 + ar\sigma^2\left(\frac{r}{\phi_s}\right)^2\right]\left(1 + ar\sigma^2\right).$$

The left-hand-side is $\left(\frac{r}{\phi_s}\left(1 + ar\sigma^2\right) - 1\right)^2$, while the right-hand-side equals

$$\left(1 + ar\sigma^2\right)^2\left(\frac{r}{\phi_s}\right)^2 - \frac{2r}{\phi_s}\left(1 + ar\sigma^2\right) + 1 + ar\sigma^2 = \left(\frac{r}{\phi_s}\left(1 + ar\sigma^2\right) - 1\right)^2 + ar\sigma^2 > LHS.$$

Thus, equation (23) holds, and the Euler equation is both necessary and sufficient for optimality (Theorem 2.3 in Chapter 1 of Fleming and Rishel (1975)).

The following sequence of argument shows that the optimal initial information rent $p_0^* > 0$, and the associated effort policy μ_t^* decreases over time.

Step 1. *From the Euler equation we show that p never changes signs. If $p(0) = p_0 > 0$, and $p(T) = 0$, but p turns negative somewhere in between, then there must exist some point $\widehat{t}$ so that*

$$p\left(\widehat{t}\right) < 0,\ p'\left(\widehat{t}\right) = 0 \text{ but } p''\left(\widehat{t}\right) \geq 0.$$

This contradicts the Euler equation (22):

$$p''\left(\widehat{t}\right)\left(\frac{1}{\phi_{\widehat{t}}}\right) = rp\left(\widehat{t}\right)\left(1 + \frac{ar\sigma^2}{1 + ar\sigma^2}\right) < 0.$$

Similarly, we can show that if $p(0) < 0$ and $p(T) = 0$, then $p < 0$ always. Therefore p never changes sign.

Step 2. *We now prove that if $p_0 > 0$ then the optimal effort policy goes down with time. From equation (22), and $\mu_t = \beta_t - p_t = -\frac{1}{\phi_t}\left(p_t' - rp_t\right) - p_t$, we have*

$$\mu_t' = -\frac{1}{\phi_t}p_t'' + \frac{1}{\phi_t}rp_t' - 2p_t' + rp_t = -rp_t\frac{ar\sigma^2}{1 + ar\sigma^2} < 0,$$

since $p_t > 0$ always. Here we also used the fact that $\left(\frac{1}{\phi_t}\right)' = 1$ from equation (20). Similarly, if $p_0 < 0$, the optimal effort policy goes up with time.

Step 3. *We show that the optimal $p_0 > 0$. First, note that a positive p_0 strictly improve the principal's value over $p_0 = 0$. When $p_0 = 0$, it follows that $p(t) = 0$, $\forall t \in [0, T]$ satisfies the Euler equation, and hence it is optimal with zero value. Now suppose that $\beta_t = \epsilon$ always where $\epsilon > 0$ is sufficiently small, so that*

$$\mu_t = \beta_t - \int_t^T \phi_s \epsilon e^{-r(s-t)} ds, \;\; p_t = \int_t^T \phi_s \epsilon e^{-r(s-t)} ds \text{ with } p_0 = \int_0^T \phi_s \epsilon e^{-r(s-t)} ds.$$

Then the value from this policy must be strictly positive, as the benefit $\int_0^T e^{-rt}\mu_t dt$ is in the order of ϵ while the cost $\int_0^T e^{-rt}\left(-\frac{1}{2}\mu_t^2 - \frac{1}{2}ar\sigma^2\beta_t^2\right) dt$ is in second order of ϵ^2. Because the constant policy might not be optimal, the policy that satisfies the Euler equation should be better, i.e., $V(p_0) > 0$. Finally, we rule out $p_0 < 0$ being optimal. To see this, we know that when $p_0 < 0$ the optimal effort policy $\{\mu_t\}$ increases over time. However,

$$\mu_T = -\frac{1}{\phi_T}\left(p_T' - rp_T\right) - p_T = -\frac{1}{\phi_T}p_T' \leq 0.$$

The last inequality follows from the fact that $p_t \leq 0$ for $t < T$ while $p_T = 0$. As a result, the optimal effort policy $\{\mu_t\}$ is always negative. Thus, if $p_0 < 0$, then the objective $\int_0^T e^{-rt}\left(\mu_t - \frac{1}{2}\mu_t^2 - \frac{1}{2}ar\sigma^2\beta_t^2\right) dt$ is negative as well.

Combining the above three steps, we complete the proof that the optimal effort policy is decreasing over time. ■

C Asymptotic Analysis for a Risk-Tolerant Agent

To achieve more analytical tractability, we perform an asymptotic analysis for agents who are sufficiently risk tolerant (relatively small a). We first establish the result for a risk-neutral agent, i.e., $a = 0$. In this case, the optimal deterministic policy obtained in Section 5.1 in the main text of the paper achieves first-best (and hence is optimal among all possible contracts).

Lemma C.1 *When $a = 0$, the deterministic policy in Proposition 3 in the main text of the paper is optimal with the first-best effort level $\mu^o = 1$. The corresponding value function, denoted by $V^o(p)$, is given by*

$$V^o(p) = -\frac{1}{2}A^o p^2 + B^o p, \tag{24}$$

where $A^o \equiv r/\phi^2$ and $B^o \equiv 1/\phi$, and the optimal information rent $\overline{p}^o$ satisfies $V_p^o(\overline{p}^o) = 0$ so that

$$\overline{p}_t^o = \overline{p}^o = \frac{B^o}{A^o} = \frac{\phi}{r}. \tag{25}$$

Proof. *Because the first-best level is achieved by the proposed contract, the result is trivial.* ■

When the agent is risk averse (i.e, $a > 0$), we consider asymptotic expansions around the benchmark case $a = 0$. More specifically, denote $a_r \equiv ar$, and we solve for the expansions in the following form:

$$V(p;a) = Q_0(p) + \sum_{i=1}^{I} a_r^i Q_i(p) + o\left(a_r^I\right), \tag{26}$$

$$\overline{p} = q_0^* - \sum_{j=1}^{J} a_r^j q_j^* + o\left(a_r^J\right), \tag{27}$$

where the highest expansion orders I and J are positive integers to be chosen later. We have $Q_0(p) = V^o(p)$ as given in equation (24), and $q_0^* = \overline{p}^o$ as given in equation (25). We solve for $Q_i(p)$ and $q_i^*(p)$ in closed-form (i.e., they are not asymptotic expansions). Also, given $V(p,a)$ it is easy to derive the optimal control pair $\{\beta(p), \sigma^p(p)\}$ based on

$$\beta = \frac{1 + p - \phi V_p}{1 + ar\sigma^2 + a^2 r^2 \sigma^2 \frac{(V_p)^2}{V_{pp}}} \text{ and } \sigma^p = -ar\sigma\beta\frac{V_p}{V_{pp}}. \tag{28}$$

The optimal deterministic contract studied in Proposition 3 in the main text of the paper is helpful in deriving Q_i's in equation (26) and q_i^*'s in equation (27). Let $\overline{p}^d \equiv \frac{B^d}{A^d} > 0$ which satisfies $V_p^d\left(\overline{p}^d\right) = 0$. By expanding both $V^d(p;a)$ and $\overline{p}^d = B^d/A^d$ in the forms of equation (26), we can compare the optimal deterministic policies to those optimal stochastic ones. We choose the expansion orders I and J to be the lowest orders so that these two expansions start to differ. For instance, we choose $J = 3$ because the asymptotic analysis reveals that the upper boundary $\overline{p}$ in the stochastic optimal contract starts to differ from the deterministic counterpart $\overline{p}^d$ in the third order a_r^3.

We first report the asymptotic expansion results for the benchmark deterministic case in which after tedious algebra we can show that:

$$V^*(p) = Q_0^*(p) + a_r Q_1^*(p) + a_r^2 Q_2^*(p) + a_r^3 Q_3^*(p) + a_r^4 Q_4^*(p) + o\left(a_r^4\right) \tag{29}$$

$$\overline{p}^* = q_0^* - q_1^* a_r - q_2^* a_r^2 - q_3^* a_r^3 - q_4^* a_r^4 + o\left(a_r^4\right) \tag{30}$$

where

$$q^{*(0)} = \frac{h}{r\sigma^2} = \overline{p}^o \tag{31a}$$

$$q^{*(1)} = \frac{h^3}{r^3\sigma^4}\left(1+\frac{r\sigma^2}{h}\right)^2 \tag{31b}$$

$$q^{*(2)} = -\frac{h^5}{r^5\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^2\left[1+\left(1+\frac{r\sigma^2}{h}\right)^2\right] \tag{31c}$$

$$q^{*(3)} = \frac{h^7}{r^7\sigma^8}\left(1+\frac{r\sigma^2}{h}\right)^2\left(\left(1+\frac{r\sigma^2}{h}\right)^2+\left[1+\left(1+\frac{r\sigma^2}{h}\right)^2\right]^2\right) \tag{31d}$$

and

$$Q^{*(0)}(p) = V^o(p) \tag{32}$$

$$Q^{*(1)}(p) = -\frac{\sigma^2}{2r}\left(1+\frac{r\sigma^2}{h}\right)^2 p^2 \tag{33}$$

$$Q^{*(2)}(p) = \frac{h^2}{2r^3}\left(1+\frac{r\sigma^2}{h}\right)^2 p^2 \tag{34}$$

$$Q^{*(3)}(p) = -\frac{h^4}{2r^5\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^2\left[1+\left(1+\frac{r\sigma^2}{h}\right)^2\right] p^2 \tag{35}$$

$$Q^{*(4)}(p) = \frac{h^6}{2r^7\sigma^4}\left(1+\frac{r\sigma^2}{h}\right)^2\left[\frac{r^2\sigma^4}{h^2}\left(2+\frac{r\sigma^2}{h}\right)^2+5\left(1+\frac{r\sigma^2}{h}\right)^2\right] p^2 \tag{36}$$

and

$$\begin{aligned}\beta^*(p) &= \left(1+\frac{r\sigma^2}{h}\right)p\left\{\begin{array}{c} 1+a_r\frac{h}{r}-a_r^2\frac{h^3}{r^3\sigma^2}\left[\frac{r^2\sigma^4}{h^2}+\left(1+\frac{r\sigma^2}{h}\right)\right] \\ +a_r^3\frac{h^5}{r^5\sigma^4}\left[\frac{r^4\sigma^8}{h^4}+\frac{r^2\sigma^4}{h^2}\left(1+\frac{r\sigma^2}{h}\right)+\left(1+\frac{r\sigma^2}{h}\right)\left(1+\left(1+\frac{r\sigma^2}{h}\right)^2\right)\right]\end{array}\right\} \\ &\equiv B^{*(0)}(p)+a_r B^{*(1)}(p)+a_r^2 B^{*(2)}(p)+a_r^3 B^{*(3)}(p).\end{aligned}$$

We now turn to the asymptotic expansion of the general model in which we consider the following approximations up to the fourth order of magnitude of a. That is,

$$V(p) = V^o(p) + a_r Q^{(1)}(p) + a_r^2 Q^{(2)}(p) + a_r^3 Q^{(3)}(p) + a_r^4 Q^{(4)}(p) + o\left(a_r^4\right), \quad (37)$$

$$\overline{p} = \overline{p}^o - a_r q^{(1)} - a_r^2 q^{(2)} - a_r^3 q^{(3)} - a_r^4 q^{(4)} + o\left(a_r^4\right). \quad (38)$$

The next proposition summarizes our results.

Proposition C.1 *When a is relatively small, we have the following approximations:*

$$\overline{p} = \overline{p}^d - a_r^3 \frac{\phi^5 \sigma^4}{r^6}\left(1+\frac{r}{\phi}\right)^3 + o\left(a_r^3\right), \quad (39)$$

$$V(\overline{p}) = V^d\left(\overline{p}^d\right) + a_r^4\left[\frac{\phi^6\sigma^6}{r^8}\left(1+\frac{r}{\phi}\right)^5 + \frac{\phi^5\sigma^6}{r^7}\left(1+\frac{r}{\phi}\right)^3\right] + o\left(a_r^4\right). \quad (40)$$

Furthermore, we show that

$$\beta(p) = \beta^d(p) - a_r^2\frac{\sigma^2}{\phi}\left(1+\frac{r}{\phi}\right)p\left(p-\overline{p}^o\right)\left[\left(p-\overline{p}^o\right)+\left(1+\frac{r}{\phi}\right)p\right] + o\left(a_r^2\right), \quad (41)$$

$$\sigma^p(p) = a_r\sigma\left(1+\frac{r}{\phi}\right)\left(1+a_r\frac{\phi\sigma^2}{r}\right)p\left(\overline{p}-p\right) + o\left(a_r^2\right). \quad (42)$$

We prove this proposition in a few steps.

In Step 1, after tedious algebra we can derive the coefficients in (37) and (38) as follows (the detailed derivation is available upon request):

$$Q^{(1)}(p) = Q^{*(1)}(p) = -\frac{\sigma^2}{2r}\left(1+\frac{r\sigma^2}{h}\right)^2 p^2 \quad (43)$$

$$Q^{(2)}(p) = Q_2^{*(2)}(p) + \frac{\sigma^6}{2h^2}\left(1+\frac{r\sigma^2}{h}\right)^2 p^2\left(p-\overline{p}^o\right)^2 \quad (44)$$

$$Q^{(3)}(p) = Q^{*(3)} - \frac{\sigma^4}{r^2}\left(1+\frac{r\sigma^2}{h}\right)^2 p^2\left[\begin{array}{c}\left(p-\overline{p}^o\right)^2+\left(1+\frac{r\sigma^2}{h}\right)p\left(p-\overline{p}^o\right)\\ -\frac{1}{2}\left(1+\frac{r\sigma^2}{h}\right)^2\left(p^2-\left(\overline{p}^o\right)^2\right)\end{array}\right] \quad (45)$$

$$Q^{(4)}(p) = Q^{*(4)}(p) - \frac{h\sigma^4}{r^3}\left(1+\frac{r\sigma^2}{h}\right)^4 p^4 + \frac{h^4}{2r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^6 p^2 + \left(p-\overline{p}^o\right)[...] \quad (46)$$

and

$$q^{(1)} = q^{*(1)} = \frac{h^3}{r^3\sigma^4}\left(1+\frac{r\sigma^2}{h}\right)^2 \quad (47)$$

$$q^{(2)} = q^{*(2)} = -\frac{h^5}{r^5\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^2\left[1+\left(1+\frac{r\sigma^2}{h}\right)^2\right] \quad (48)$$

$$q^{(3)} = q^{*(3)} + \frac{h^5}{r^6\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^3 \quad (49)$$

Therefore, $\overline{p}-\overline{p}^{*}=-a_r^3\left(q^{(3)}-q^{*(3)}\right)=-a_r^3\frac{h^5}{r^6\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^3+o\left(a_r^3\right)$, resulting in (39).

In Step 2, we derive (40). Note that

$$\begin{aligned}
V\left(\overline{p}\right) &= V^o\left(\overline{p}^o\right)-\frac{1}{2}A^o\left(a_r q^{(1)}+a_r^2 q^{(2)}+a_r^3 q^{(3)}\right)^2 \\
&\quad +a_r Q^{*(1)}\left(\overline{p}\right)+a_r^2\left[Q^{*(2)}\left(\overline{p}\right)+\frac{\sigma^6}{2h^2}\left(1+\frac{r\sigma^2}{h}\right)^2\overline{p}^2\left(\overline{p}-\overline{p}^o\right)^2\right] \\
&\quad +a_r^3\left[Q^{*(3)}\left(\overline{p}\right)+\frac{\sigma^4}{r^2}\left(1+\frac{r\sigma^2}{h}\right)^3\overline{p}^2\left(\overline{p}-\overline{p}^o\right)\right] \\
&\quad +a_r^4\left[Q^{*(4)}\left(\overline{p}\right)-\frac{h\sigma^4}{r^3}\left(1+\frac{r\sigma^2}{h}\right)^4\overline{p}^4+\frac{h^4}{2r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^6\overline{p}^2\right]+o\left(a_r^4\right) \\
&= V^o\left(\overline{p}^o\right)-\frac{1}{2}A^o\left(a_r q^{*(1)}+a_r^2 q^{*(2)}+a_r^3 q^{*(3)}+a_r^3\frac{h^5}{r^6\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^3\right)^2 \\
&\quad +a_r\left[Q^{*(1)}\left(\overline{p}^*\right)+a_r^3\frac{h^6}{r^8\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^5\right] \\
&\quad +a_r^2\left[Q^{*(2)}\left(\overline{p}^*\right)+\frac{\sigma^6}{2h^2}\left(1+\frac{r\sigma^2}{h}\right)^2\left(\overline{p}^o\right)^2 a_r^2\left(q^{*(1)}\right)^2\right] \\
&\quad +a_r^3\left[Q^{*(3)}\left(\overline{p}^*\right)+\frac{\sigma^4}{r^2}\left(1+\frac{r\sigma^2}{h}\right)^3\left(\overline{p}^o\right)^2\left(-a_r q^{*(1)}\right)\right] \\
&\quad +a_r^4\left[Q^{*(4)}\left(\overline{p}\right)-\frac{h\sigma^4}{r^3}\left(1+\frac{r\sigma^2}{h}\right)^4\left(\overline{p}^o\right)^4+\frac{h^4}{2r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^6\left(\overline{p}^o\right)^2\right]+o\left(a_r^4\right) \\
&= V^*\left(p^*\right)+a_r^4\left[\begin{array}{c} -A^o q^{*(1)}\frac{h^5}{r^6\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)+\frac{h^6}{r^8\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^5+\frac{\sigma^6}{2h^2}\left(1+\frac{r\sigma^2}{h}\right)^2\left(\overline{p}^o\right)^2\left(q^{*(1)}\right)^2 \\ -\frac{\sigma^4}{r^2}\left(1+\frac{r\sigma^2}{h}\right)^3\left(\overline{p}^o\right)^2 q^{*(1)}-\frac{h\sigma^4}{r^3}\left(1+\frac{r\sigma^2}{h}\right)^4\left(\overline{p}^o\right)^4+\frac{h^4}{2r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^6\left(\overline{p}^o\right)^2 \end{array}\right]
\end{aligned}$$

The term in the bracket is

$$\begin{aligned}
&\left[\begin{array}{l} -A^o q^{*(1)} \frac{h^5}{r^6\sigma^6}\left(1+\frac{r\sigma^2}{h}\right) + \frac{h^6}{r^8\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^5 + \frac{\sigma^6}{2h^2}\left(1+\frac{r\sigma^2}{h}\right)^2 (\overline{p}^o)^2 \left(q^{*(1)}\right)^2 \\ -\frac{\sigma^4}{r^2}\left(1+\frac{r\sigma^2}{h}\right)^3 (\overline{p}^o)^2 q^{*(1)} - \frac{h\sigma^4}{r^3}\left(1+\frac{r\sigma^2}{h}\right)^4 (\overline{p}^o)^4 + \frac{h^4}{2r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^6 (\overline{p}^o)^2 \end{array}\right] \\
= \; & -\frac{h^6}{r^8\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^3 + \frac{h^6}{r^8\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^5 + \frac{1}{2}\frac{h^6}{r^8\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^6 \\
& -\frac{h^5}{r^7\sigma^4}\left(1+\frac{r\sigma^2}{h}\right)^5 - \frac{h^5}{r^7\sigma^4}\left(1+\frac{r\sigma^2}{h}\right)^4 + \frac{h^6}{2r^8\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^6 \\
= \; & \frac{h^6}{r^8\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^3\left[\left(1+\frac{r\sigma^2}{h}\right)^3 + \left(1+\frac{r\sigma^2}{h}\right)^2 - 1\right] - \frac{h^5}{r^7\sigma^4}\left(1+\frac{r\sigma^2}{h}\right)^4\left(2+\frac{r\sigma^2}{h}\right) \\
= \; & \frac{h^6}{r^8\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^3\left[\left(1+\frac{r\sigma^2}{h}\right)^3 + \frac{r\sigma^2}{h}\left(2+\frac{r\sigma^2}{h}\right)\right] - \frac{h^5}{r^7\sigma^4}\left(1+\frac{r\sigma^2}{h}\right)^4\left(2+\frac{r\sigma^2}{h}\right) \\
= \; & \frac{h^6}{r^8\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^6 + \frac{h^5}{r^7\sigma^4}\left(1+\frac{r\sigma^2}{h}\right)^3\left(2+\frac{r\sigma^2}{h}\right) - \frac{h^5}{r^7\sigma^4}\left(1+\frac{r\sigma^2}{h}\right)^4\left(2+\frac{r\sigma^2}{h}\right) \\
= \; & \frac{h^6}{r^8\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^6 - \frac{h^4}{r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^3\left(2+\frac{r\sigma^2}{h}\right) \\
= \; & \frac{h^6}{r^8\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^6 - \frac{h^4}{r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^4 - \frac{h^4}{r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^3 \\
= \; & \frac{h^4}{r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^4\left[\frac{h^2}{r^2\sigma^4}\left(1+\frac{r\sigma^2}{h}\right)^2 - 1\right] - \frac{h^4}{r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^3 \\
= \; & \frac{h^4}{r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^4\left[\frac{h^2}{r^2\sigma^4} + \frac{2h}{r\sigma^2}\right] - \frac{h^4}{r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^3 \\
= \; & \frac{h^4}{r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^4\left[\frac{h^2}{r^2\sigma^4} + \frac{h}{r\sigma^2}\right] + \frac{h^4}{r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^3\left(1+\frac{h}{r\sigma^2}\right) - \frac{h^4}{r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^3 \\
= \; & \frac{h^4}{r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^5 \frac{h^2}{r^2\sigma^4} + \frac{h^4}{r^6\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^3 \frac{h}{r\sigma^2} \\
= \; & \frac{h^6}{r^8\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^5 + \frac{h^5}{r^7\sigma^4}\left(1+\frac{r\sigma^2}{h}\right)^3
\end{aligned}$$

In Step 3, we derive (41). Note that

$$\frac{1+p-\frac{h}{\sigma^2}V_p}{1+a_r\sigma^2+a_r^2\sigma^2\frac{V_p^2}{V_{pp}}} = \frac{1+p-\frac{h}{\sigma^2}\left[V_p^o+a_rQ_p^{(1)}+a_r^2Q_p^{(2)}+a_r^3Q_p^{(3)}+a_r^4Q_p^{(4)}\right]}{1+a_r\sigma^2+a_r^2\sigma^2\left[\frac{\left(V_p^o\right)^2}{V_{pp}^o}+a_rR^{(1)}+a_r^2R^{(2)}\right]}$$

$$\begin{aligned}
= & \left[\left(1+p-\frac{hV_p^o}{\sigma^2}\right)-\frac{hQ_p^{(1)}}{\sigma^2}a_r-\frac{hQ_p^{(2)}}{\sigma^2}a_r^2-\frac{hQ_p^{(3)}}{\sigma^2}a_r^3+\cdots\right] \\
& \times\left[1-\sigma^2a_r+\left(\sigma^4-\frac{\sigma^2\left(V_p^o\right)^2}{V_{pp}^o}\right)a_r^2+\left(-\sigma^6+\frac{2\sigma^4\left(V_p^o\right)^2}{V_{pp}^o}-R^{(1)}\sigma^2\right)a_r^3+\cdots\right] \\
= & \left(1+p-\frac{hV_p^o}{\sigma^2}\right)-a_r\left[\left(1+p-\frac{hV_p^o}{\sigma^2}\right)\sigma^2+\frac{hQ_p^{(1)}}{\sigma^2}\right] \\
& +a_r^2\left[\left(1+p-\frac{hV_p^o}{\sigma^2}\right)\left(\sigma^4-\frac{\sigma^2\left(V_p^o\right)^2}{V_{pp}^o}\right)+\sigma^2\frac{hQ_p^{(1)}}{\sigma^2}-\frac{hQ_p^{(2)}}{\sigma^2}\right] \\
& +a_r^3\left[\begin{array}{c}\left(1+p-\frac{hV_p^o}{\sigma^2}\right)\left(-\sigma^6+\frac{2\sigma^4\left(V_p^o\right)^2}{V_{pp}^o}-R^{(1)}\sigma^2\right)-\frac{hQ_p^{(1)}}{\sigma^2}\left(\sigma^4-\frac{\sigma^2\left(V_p^o\right)^2}{V_{pp}^o}\right) \\ +\sigma^2\frac{hQ_p^{(2)}}{\sigma^2}-\frac{hQ_p^{(3)}}{\sigma^2}\end{array}\right] \\
= & B^{*(0)}+a_rB^{*(1)}+a_r^2B^{*(2)}+a_r^3B^{*(3)}+a_r^2\left[\left(1+p-\frac{hV_p^o}{\sigma^2}\right)\left(-\frac{\sigma^2\left(V_p^o\right)^2}{V_{pp}^o}\right)-\frac{h\left(Q_p^{(2)}-Q_p^{*(2)}\right)}{\sigma^2}\right] \\
& +a_r^3\left[\begin{array}{c}\left(1+p-\frac{hV_p^o}{\sigma^2}\right)\left(\frac{2\sigma^4\left(V_p^o\right)^2}{V_{pp}^o}-R^{(1)}\sigma^2\right)+\frac{hQ_p^{(1)}}{\sigma^2}\frac{\sigma^2\left(V_p^o\right)^2}{V_{pp}^o} \\ +\left(\sigma^2\frac{hQ_p^{(2)}}{\sigma^2}-\frac{hQ_p^{(3)}}{\sigma^2}\right)-\left(\sigma^2\frac{hQ_p^{*(2)}}{\sigma^2}-\frac{hQ_p^{*(3)}}{\sigma^2}\right)\end{array}\right] \\
= & B^{*(0)}+a_rB^{*(1)}+a_r^2B^{*(2)}+a_r^3B^{*(3)}-a_r^2\frac{\sigma^4}{h}\left(1+\frac{r\sigma^2}{h}\right)p\left(p-\overline{p}^o\right)\left[\left(p-\overline{p}^o\right)+\left(1+\frac{r\sigma^2}{h}\right)p\right] \\
& +a_r^3\left[\begin{array}{c}-\left(1+\frac{r\sigma^2}{h}\right)p\left(\frac{2r\sigma^6}{h^2}\left(p-\overline{p}^o\right)^2+R^{(1)}\sigma^2\right)+\frac{\sigma^6}{h}\left(1+\frac{r\sigma^2}{h}\right)^2p\left(p-\overline{p}^o\right)^2 \\ -\frac{\sigma^4}{r}\left(1+\frac{r\sigma^2}{h}\right)^3p^3+\frac{h\sigma^2}{r^2}\left(1+\frac{r\sigma^2}{h}\right)^2p\left(p-\overline{p}^o\right)R^{(4)}\left(p\right)\end{array}\right]
\end{aligned}$$

where we used

$$R^{(1)}\left(p\right)=-\frac{\sigma^2}{r}\left(1+\frac{r\sigma^2}{h}\right)^2\left(p^2-\left(\overline{p}^o\right)^2\right)$$

$$Q_p^{(2)}\left(p\right)=Q_p^{*(2)}\left(p\right)+\frac{\sigma^6}{h^2}\left(1+\frac{r\sigma^2}{h}\right)^2p\left(p-\overline{p}^o\right)\left(2p-\overline{p}^o\right)$$

$$-\sigma^2\frac{hQ_p^{(2)}}{\sigma^2}+\frac{hQ_p^{(3)}}{\sigma^2}=\left[-\sigma^2\frac{hQ_p^{*(2)}}{\sigma^2}+\frac{hQ_p^{*(3)}}{\sigma^2}\right]+\frac{\sigma^4}{r}\left(1+\frac{r\sigma^2}{h}\right)^3p^3-\frac{h\sigma^2}{r^2}\left(1+\frac{r\sigma^2}{h}\right)^2p\left(p-\overline{p}^o\right)R^{(4)}\left(p\right)$$

where

$$R^{(4)}\left(p\right)\equiv\left(\frac{r^2\sigma^4}{h^2}+\frac{r\sigma^2}{h}+6\right)p-\left(3+2\frac{r^2\sigma^4}{h^2}+2\frac{r\sigma^2}{h}\right)\overline{p}^o$$

Therefore,

$$\begin{aligned}\beta(\overline{p}) &= \beta^*(\overline{p}^*) + a_r^3\left[\frac{\sigma^4}{h}\left(1+\frac{r\sigma^2}{h}\right)^2(\overline{p}^o)^2 q^{(1)} - \frac{\sigma^4}{r}\left(1+\frac{r\sigma^2}{h}\right)^3(\overline{p}^o)^3\right] \\ &= \beta^*(\overline{p}^*) + a_r^3\left(1+\frac{r\sigma^2}{h}\right)^3\frac{h^4}{r^5\sigma^4}\end{aligned}$$

In Step 4, we derive (42). Note that

$$\beta(p) = B^{*(0)} + a_r B^{*(1)} + a_r^2 B^{*(2)} - a_r^2\frac{\sigma^4}{h}\left(1+\frac{r\sigma^2}{h}\right)p\,(p-\overline{p}^o)\left[(p-\overline{p}^o)+\left(1+\frac{r\sigma^2}{h}\right)p\right] + o\left(a_r^2\right)$$

and

$$\begin{aligned}\frac{V_p}{V_{pp}} &= \frac{V_p^o + a_r Q_p^{(1)} + a_r^2 Q_p^{(2)} + a_r^3 Q_p^{(3)} + a_r^4 Q_p^{(4)} + O\left(a_r^5\right)}{V_{pp}^o + a_r Q_{pp}^{(1)} + a_r^2 Q_{pp}^{(2)} + a_r^3 Q_{pp}^{(3)} + a_r^4 Q_{pp}^{(4)} + O\left(a_r^5\right)} \\ &= \frac{V_p^o/V_{pp}^o + a_r Q_p^{(1)}/V_{pp}^o + a_r^2 Q_p^{(2)}/V_{pp}^o + a_r^3 Q_p^{(3)}/V_{pp}^o + a_r^4 Q_p^{(4)}/V_{pp}^o + O\left(a_r^5\right)}{1 + a_r Q_{pp}^{(1)}/V_{pp}^o + a_r^2 Q_{pp}^{(2)}/V_{pp}^o + a_r^3 Q_{pp}^{(3)}/V_{pp}^o + a_r^4 Q_{pp}^{(4)}/V_{pp}^o + O\left(a_r^5\right)} \\ &= \left[V_p^o/V_{pp}^o + a_r Q_p^{(1)}/V_{pp}^o + a_r^2 Q_p^{(2)}/V_{pp}^o + \cdots\right] \\ &\quad \times\left[1 - a_r Q_{pp}^{(1)}/V_{pp}^o + \left(\left(Q_{pp}^{(1)}/V_{pp}^o\right)^2 - Q_{pp}^{(2)}/V_{pp}^o\right)a_r^2 + \cdots\right] \\ &= \frac{V_p^o}{V_{pp}^o} + a_r\left[\frac{Q_p^{(1)}}{V_{pp}^o} - \frac{V_p^o}{\left(V_{pp}^o\right)^2}Q_{pp}^{(1)}\right] + a_r^2\left[\frac{Q_p^{(2)}}{V_{pp}^o} - \frac{Q_p^{(1)}Q_{pp}^{(1)}}{\left(V_{pp}^o\right)^2} + \frac{V_p^o}{V_{pp}^o}\left(\left(\frac{Q_{pp}^{(1)}}{V_{pp}^o}\right)^2 - \frac{Q_{pp}^{(2)}}{V_{pp}^o}\right)\right] + \cdots \\ &= (p-\overline{p}^o) - \frac{a_r}{A^o}\left[p-(p-\overline{p}^o)\right]Q_{pp}^{(1)} + a_r^2\left[\frac{Q_p^{(2)} - (p-\overline{p}^o)\,Q_{pp}^{(2)}}{V_{pp}^o} - \overline{p}^o\left(\frac{Q_{pp}^{(1)}}{V_{pp}^o}\right)^2\right] \\ &= (p-\overline{p}^o) - a_r\frac{h^3}{r^2\sigma^6}Q_{pp}^{(1)} - a_r^2\left[\frac{h^2}{r\sigma^4}\left(Q_p^{(2)} - (p-\overline{p}^o)\,Q_{pp}^{(2)}\right) - \frac{h}{r\sigma^2}\frac{h^4}{r^2\sigma^8}\left(Q_{pp}^{(1)}\right)^2\right] \\ &= (p-\overline{p}^o) - a_r\frac{h^3}{r^2\sigma^6}\left[-\frac{\sigma^2}{r}\left(1+\frac{r\sigma^2}{h}\right)^2\right] \\ &\quad - a_r^2\left[\frac{h^2}{r\sigma^4}\left[\frac{h^3}{r^4\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^2 - \frac{\sigma^6}{h^2}\left(1+\frac{r\sigma^2}{h}\right)^2(p-\overline{p}^o)^2(4p-\overline{p}^o)\right] - \frac{h^5}{r^5\sigma^6}\left(1+\frac{r\sigma^2}{h}\right)^4\right] \\ &= (p-\overline{p}^o) + a_r\frac{h^3}{r^3\sigma^4}\left(1+\frac{r\sigma^2}{h}\right)^2 + a_r^2\left(1+\frac{r\sigma^2}{h}\right)^2\left[\frac{h^4}{r^4\sigma^4}\left(2+\frac{r\sigma^2}{h}\right) + \frac{\sigma^2}{r}(p-\overline{p}^o)^2(4p-\overline{p}^o)\right]\end{aligned}$$

where we used

$$
\begin{aligned}
Q^{(2)}(p) &= \frac{h^2}{2r^3}\left(1+\frac{r\sigma^2}{h}\right)^2 p^2+\frac{\sigma^6}{2h^2}\left(1+\frac{r\sigma^2}{h}\right)^2 p^2\left(p-\overline{p}^o\right)^2 \\
Q_p^{(2)}(p) &= \frac{h^2}{r^3}\left(1+\frac{r\sigma^2}{h}\right)^2 p+\frac{\sigma^6}{h^2}\left(1+\frac{r\sigma^2}{h}\right)^2 p\left(p-\overline{p}^o\right)\left(2p-\overline{p}^o\right) \\
Q_{pp}^{(2)}(p) &= \frac{h^2}{r^3}\left(1+\frac{r\sigma^2}{h}\right)^2+\frac{\sigma^6}{h^2}\left(1+\frac{r\sigma^2}{h}\right)^2\left[\left(p-\overline{p}^o\right)\left(4p-\overline{p}^o\right)+p\left(2p-\overline{p}^o\right)\right] \\
Q_p^{(2)}-\left(p-\overline{p}^o\right)Q_{pp}^{(2)} &= \frac{h^3}{r^4\sigma^2}\left(1+\frac{r\sigma^2}{h}\right)^2-\frac{\sigma^6}{h^2}\left(1+\frac{r\sigma^2}{h}\right)^2\left(p-\overline{p}^o\right)^2\left(4p-\overline{p}^o\right)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sigma^p &= -a_r\sigma\beta\frac{V_p}{V_{pp}} \\
&= -a_r\sigma\left[B^{*(0)}+a_rB^{*(1)}+a_r^2B^{*(2)}-a_r^2\frac{\sigma^4}{h}\left(1+\frac{r\sigma^2}{h}\right)p\left(p-\overline{p}^o\right)\left[\left(p-\overline{p}^o\right)+\left(1+\frac{r\sigma^2}{h}\right)p\right]\right] \\
&\quad\times\left[\left(p-\overline{p}^o\right)+a_r\frac{h^3}{r^3\sigma^4}\left(1+\frac{r\sigma^2}{h}\right)^2+a_r^2\left(1+\frac{r\sigma^2}{h}\right)^2\left[\frac{h^4}{r^4\sigma^4}\left(2+\frac{r\sigma^2}{h}\right)+\frac{\sigma^2}{r}\left(p-\overline{p}^o\right)^2\left(4p-\overline{p}^o\right)\right]\right] \\
&= -a_r\sigma\left\{\begin{array}{c} B^{*(0)}\left(p-\overline{p}^o\right)+a_r\left[B^{*(0)}\frac{h^3}{r^3\sigma^4}\left(1+\frac{r\sigma^2}{h}\right)^2+B^{*(1)}\left(p-\overline{p}^o\right)\right] \\ +a_r^2\left[B^{*(0)}\left(1+\frac{r\sigma^2}{h}\right)^2\left[\frac{h^4}{r^4\sigma^4}\left(2+\frac{r\sigma^2}{h}\right)+\frac{\sigma^2}{r}\left(p-\overline{p}^o\right)^2\left(4p-\overline{p}^o\right)\right]\right]+\cdots \end{array}\right\} \\
&= -a_r\sigma\left\{\begin{array}{c} B^{*(0)}\left(\left(p-\overline{p}\right)-a_rq^{(1)}+\cdots\right) \\ +a_r\left[B^{*(0)}\frac{h^3}{r^3\sigma^4}\left(1+\frac{r\sigma^2}{h}\right)^2+B^{*(1)}\left(\left(p-\overline{p}\right)-a_rq^{(1)}+\cdots\right)\right] \\ +a_r^2\left[B^{*(0)}\left(1+\frac{r\sigma^2}{h}\right)^2\left[\frac{h^4}{r^4\sigma^4}\left(2+\frac{r\sigma^2}{h}\right)+\frac{\sigma^2}{r}\left(p-\overline{p}^o\right)^2\left(4p-\overline{p}^o\right)\right]\right]+\cdots \end{array}\right\} \\
&= -a_r\sigma\left\{\left(1+\frac{r\sigma^2}{h}\right)p\left(p-\overline{p}\right)+a_r\left(1+\frac{r\sigma^2}{h}\right)\frac{h}{r}p\left(p-\overline{p}\right)+\cdots\right\} \\
&= a_r\sigma\left(1+\frac{r\sigma^2}{h}\right)\left(1+a_r\frac{h}{r}+\cdots\right)p\left(\overline{p}-p\right)+o\left(a_r^2\right)
\end{aligned}
$$

D Numerical Algorithm in Solving for Value Function

D.1 Numerical Integration Method

The key ODE is

$$rV = \frac{1}{2}\frac{(1+p-\phi V_p)^2}{1+ar\sigma^2+a^2r^2\sigma^2\frac{V_p^2}{V_{pp}}} - p - \frac{1}{2}p^2 + V_p(\phi+r)\,p, \tag{50}$$

where boundary conditions are: $V(0)=0$ and $V'(0)=1/\phi$. Furthermore, at $\overline{p}$, $V'(\overline{p})=0$.

Define $U(p)\equiv V(p)-\frac{p}{\phi}$. Note that $U(0)=U_p(0)=0$, $U_p(p)=V_p(p)-1/\phi$, and $U_{pp}(p)=V_{pp}(p)$, the ODE in Eq. (50) implies that[2]

$$\left(rU+\frac{1}{2}p^2-U_p(\phi+r)\right)ar\sigma^2\left(U_{pp}+ar(U_p+1/\phi)^2\right)=\left(\frac{1}{2}\phi^2U_p^2-rU+rpU_p\right)U_{pp}. \tag{51}$$

We solve the ODE in Eq. (51) using numerical integration. The algorithm is the following:

1. Given a guess of $\overline{p}$, create a equally spaced grid $\{p_i\}_{i=1}^{n+2}$ of the interval $[0,\overline{p}]$. Let $\Delta_p \equiv \frac{p_{n+2}-p_1}{n+1}$.

2. Let $\{x_i\}_{i=1}^{n+2}$ denote the second-order derivative of $U(p)$ at each grid point. That is, $x_i = U_{pp}(p_i)$.

3. Given the boundary condition $U_p(0)=0$, by numerical integration, we can solve for the first-order derivative of $U(p)$ at each grid point, denoted by $y_i=U_p(p_i)$. That is, for $i\geq 2$

$$y_i=\int_0^{p_i}U_{pp}(p)\,dp=\left(\frac{1}{2}x_1+\sum_{j=2}^{i-1}x_j+\frac{1}{2}x_i\right)\Delta_p$$

4. Similarly, for each grid point p_i, we can use numerical integration to solve for the value function at p_i, denoted by z_i, given the boundary condition $U(0)=0$ and the numerical

[2] The derivation is below:

$$rV+p+\frac{1}{2}p^2-V_p(\phi+r)\,p=\frac{1}{2}\frac{(1+p-\phi V_p)^2\,V_{pp}}{(1+ar\sigma^2)\,V_{pp}+a^2r^2\sigma^2V_p^2},$$

or

$$r(U+p/\phi)+p+\frac{1}{2}p^2-(U_p+1/\phi)(\phi+r)\,p=\frac{1}{2}\frac{(1+p-\phi(U_p+1/\phi))^2\,U_{pp}}{(1+ar\sigma^2)\,U_{pp}+a^2r^2\sigma^2(U_p+1/\phi)^2}=rU+\frac{1}{2}p^2-U_p(\phi+r)\,p,$$

or

$$\begin{aligned}&\left(rU+\frac{1}{2}p^2-U_p(\phi+r)\right)ar\sigma^2\left(U_{pp}+ar(U_p+1/\phi)^2\right)\\=&\left[\frac{1}{2}(1+p-\phi(U_p+1/\phi))^2-\left(rU+\frac{1}{2}p^2-U_p(\phi+r)\,p\right)\right]U_{pp}\\=&\left[\frac{1}{2}\phi^2U_p^2-rU+rpU_p\right]U_{pp}.\end{aligned}$$

solution of $U_p(p)$. That is, for $i \geq 2$

$$z_i = \int_0^{p_i} U_p(p)\, dp = \left(\frac{1}{2} y_1 + \sum_{j=2}^{i-1} y_j + \frac{1}{2} y_i \right) \Delta_p$$

5. As a result of Steps 2-4, we can express the value function and its first- and second-order derivatives in terms of $\{x_i\}_{i=1}^{n+2}$ given the conjectured $\overline{p}$. We then substitute the expressions into the ODE in Eq. (51) and solve for $\{x_i\}_{i=1}^{n+2}$ to minimize the root mean squared errors, subject to the following constraints

$$\int_0^{\overline{p}} U_{pp}(p)\, dp = \int_0^{\overline{p}} V_{pp}(p)\, dp = V_p(\overline{p}) - V_p(0) = -\frac{1}{\phi},$$
$$\left(1 + ar\sigma^2\right) U_{pp} + a^2 r^2 \sigma^2 \left(U_p + 1/\phi\right)^2 = \left(1 + ar\sigma^2\right) V_{pp} + a^2 r^2 \sigma^2 V_p^2 < 0,$$
$$U_{pp} = V_{pp} < 0.$$

6. We repeat Steps 1-5 for a wide range of conjectured values for $\overline{p}$, and then select the optimal value $\overline{p}^*$ to achieve the maximal value function at the conjectured value $\overline{p}$, i.e., $V(\overline{p}^*) = U(\overline{p}^*) + \frac{\overline{p}^*}{\phi} = \max_{\overline{p}} V(\overline{p})$.

D.2 MATLAB Implementation

In implementing the algorithm discussed in the previous section in Matlab, we develop the following Matlab programs:

- "**maincode.m**" — The main Matlab program that should be run first. In this program, we include a grid of possible values of $\overline{p}$, and then for each possible guess, we solve ODE (50) by minimizing the objective function coded in "obj_Upp.m" in the presence of the constraints (see Step 5 in the algorithm in the previous section). In particular, the non-linear constraint is coded in "**nlconfun.m**". The equilibrium outcome under the optimal contract is then saved and used in simulating the model, which is coded in "**simu.m**".
- "**obj_Upp.m**" — This program calculates the objective function used in solving the ODE.
- "**nlconfun.m**" — This program specifies the nonlinear constraint used in solving the ODE.
- "**simu.m**" — This program simulates the model for 10,000 rounds and each round has 250 periods.

D.2.1 "maincode.m"

```
% main program for "Optimal Long-term Contracting with Learning"
    format long;
    baseline_spec=1; % 1, if baseline specification; 0, otherwise
    if baseline_spec=1
    % parameter specification under baseline specification
    r=0.5; a=1; sig=8; phi=0.5;
    filename='.\results_bm.mat';
    else
    % parameter specification under baseline specification
    r=0.5; a=1; sig=8; phi=0.75;
    filename='.\results_phi.mat';
    end
    h=phi*sig^2;
    % parameter condition
    ParamCond=phi/r-a*(2*(phi/r+1)^3-phi*sig^2);
    if ParamCond<=0
    error('choose different parameter values to satisfy the condition');
    end
    % Holmstrom-Milgrom benchmark case
    V_HM=1/(2*r*(1+a*r*sig^2));
    % Deterministic Case
    Am=((2*h+r*sig^2)*a*r+r+sqrt((2*h+r*sig^2)^2*a^2*r^2 ...
    +2*sig^2*((h/sig^2+r)^2+h^2/sig^4)*a*r+r^2))*sig^4/(2*h^2);
    Bm=sig^2/h;
    pbarm=Bm/Am;
    num=201-2; % num+2 is the number of grid points for "p"
    pm_vec=linspace(0,pbarm,num+2);
    Vm_vec=-1/2*Am*pm_vec.^2+Bm*pm_vec;
    % equilibrium outcomes in the general case
    pbar_num=25; % number of grid points for "pbar"
    V_mat=zeros(num+2,pbar_num); % value function
```

```
Vp_mat=zeros(num+2,pbar_num); % 1st-order derivative of value function
Vpp_mat=zeros(num+2,pbar_num); % 2nd-order derivative of value function
pgrid_mat=zeros(num+2,pbar_num); % pbar
beta_mat=zeros(num+2,pbar_num); % incentives as a function of "p"
sigP_mat=zeros(num+2,pbar_num); % incentive diffusion as a function of "p"
for i=1:pbar_num
pbar=pbarm-0.0000005*i;
pgrid=linspace(0,pbar,num+2);
dp=pbar/(num+1);
if i==1
Upp_init=-ones(1,num+2)*Am;
else
Upp_init=interp1(pgrid_mat(:,i-1),Vpp_mat(:,i-1),pgrid);
end
options = optimset('MaxFunEvals',8000000,'MaxIter',800000);
lb=ones(num+2,1)*(-100000);
ub=zeros(num+2,1);
Aeq=[1 ones(1,num)*2 1]*dp*0.5;
beq=-1/phi;
objfun=@(Upp) obj_Upp(Upp,r,a,sig,phi,num,pgrid,dp);
nlcon=@(Upp) nlconfun(Upp,r,a,sig,phi,num,dp);
[Upp,fval]=fmincon(objfun,Upp_init,[],[],Aeq,beq,lb,ub,nlcon,options);
% smoothing Upp to get rid of oscillations near pbar
coef_Upp=polyfit(pgrid,Upp,2);
Upp_smooth=polyval(coef_Upp,pgrid);
plot([Upp',Upp_smooth']);
% adjustment of smoothed Upp to satisfy the constraint Aeq*Upp'=beq;
Upp_diff=(Aeq*Upp_smooth'-beq)/sum(Aeq);
Upp=Upp_smooth-Upp_diff;
Vpp=Upp;
Up=[0 cumsum(Upp(1:(num+1)))*0.5*dp+cumsum(Upp(2:(num+2)))*0.5*dp];
Vp=1/phi+Up;
```

```
U=[0 cumsum(Up(1:(num+1)))*0.5*dp+cumsum(Up(2:(num+2)))*0.5*dp];
V=U+pgrid/phi;
pgrid_mat(:,i)=pgrid;
Vpp_mat(:,i)=Upp;
Vp_mat(:,i)=Vp;
V_mat(:,i)=V;
beta_mat(:,i)=(1+pgrid-phi*Vp)./(1+a*r*sig^2+a^2*r^2*sig^2*Vp.^2./Vpp);
sigP_mat(:,i)=-a*r*sig*beta_mat(:,i)'.*Vp./Vpp;
% [i,fval,num,V(end),Vm_vec(end)]
save(filename,'r','a','sig','h','phi','V_mat','Vpp_mat','pgrid_mat','beta_mat','sigP_mat','Vm_vec');
end
```

D.2.2 "obj_Upp.m"

```
function FF = obj_Upp(Upp, r, a, sig, phi,num,pgrid,dp)
    % objective funciton for solving Upp, derived from the ODE of U function
    % NOTE: U=V-p/phi
    xn=Upp;
    yn=[0 cumsum(xn(1:(num+1)))*0.5*dp+cumsum(xn(2:(num+2)))*0.5*dp];
    zn=[0 cumsum(yn(1:(num+1)))*0.5*dp+cumsum(yn(2:(num+2)))*0.5*dp];
    rs=(0.5*phi^2*yn.^2-r*zn+r*pgrid.*yn).*xn;
    ls=a*r*sig^2*(xn+a*r*(1/phi+yn).^2).*(r*zn+0.5*pgrid.^2-(phi+r)*pgrid.*yn);
    FF=(rs-ls)*(rs-ls)';
```

D.2.3 "nlconfun.m"

```
function [FF oeq]= nlconfun(Upp, r, a, sig,phi,num,dp)
    % non-linear constraint in solving the ODE of U function
    % NOTE: U=V-p/phi
    xn=Upp;
    yn=[0 cumsum(xn(1:(num+1)))*0.5*dp+cumsum(xn(2:(num+2)))*0.5*dp];
    FF=(1+a*r*sig^2)*xn+a^2*r^2*sig^2*(1/phi+yn).^2;
    oeq=[];
```

D.2.4 "simu.m"

```
% simulations
clear; close all; clc;
% r=0.5; a=1; sig=8; phi=0.5;
filename='.\results_bm.mat';
load(filename);
num=size(V_mat,1)-2;
id=find(V_mat(end,:)==max(V_mat(end,:)));
% stochastic case
beta2=beta_mat(:,id);
sigP2=sigP_mat(:,id);
pgrid2=pgrid_mat(:,id);
dp2=pgrid2(2)-pgrid2(1);
mu2=beta2-pgrid2;
pbar=pgrid2(end);
simuNO=10000;
simuT=250;
dt=1/12;
p2_drift=(phi+r)*pgrid2+beta2.*(a*r*sig*sigP2-phi);
p2_diffusion=sigP2;
p_simu=zeros(simuT,simuNO);
beta_simu=zeros(simuT,simuNO);
sigP_simu=zeros(simuT,simuNO);
for i=1:simuNO
p_prev=pbar;
id_prev=num+2;
for j=1:simuT
p=p_prev+p2_drift(id_prev)*dt+p2_diffusion(id_prev)*normrnd(0,sqrt(dt));
if p>=pbar
p=pbar;
p_prev=p;
id_prev=num+2;
```

```
elseif p<=0
p=0;
p_prev=p;
id_prev=1;
else
p_prev=p;
id_prev=find(abs(pgrid2-p)==min(abs(pgrid2-p)));
id_prev=max(id_prev);
end
beta_simu(j,i)=beta2(id_prev);
sigP_simu(j,i)=sigP2(id_prev);
p_simu(j,i)=pgrid2(id_prev);
mu_simu(j,i)=mu2(id_prev);
end
end
lw=2;
fs=12;
f1=figure;
subplot(3,1,1);
plot(mu_simu);
xlabel('t','FontSize',fs);
title('Panel A: Simulated Effort Paths','FontSize',fs);
subplot(3,1,2);
plot(mean(mu_simu'),'LineWidth',lw)
xlabel('t','FontSize',fs);
title('Panel B: Mean of Simulated Effort Paths','FontSize',fs);
subplot(3,1,3);
plot(sqrt(var(mu_simu')),'LineWidth',lw);
xlabel('t','FontSize',fs);
title('Panel C: Volatility of Simulated Effort Paths','FontSize',fs);
% print -f1 -dpsc2 '.\mu_simu.eps';
```